Mechanics

Kinematics

$a(t) = \text{const.}$

$$v(t) = \int_{t=0}^{t} a \, dt' = \left[a \cdot t'\right]_0^t + C = a \cdot t - a \cdot 0 + C = a \cdot t + v_0$$

$$x(t) = \int_{t=0}^{t} v \cdot dt' = \int_0^t (a \cdot t' + V_0) = \left[\frac{1}{2} a t'^2 + V_0 t'\right]_0^t = \frac{1}{2} a t^2 + V_0 t + C$$

$$= \frac{1}{2} a t^2 + V_0 t + X_0$$

Vectors

position

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

velocity

$$\vec{v}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix}$$

acceleration

$$\vec{a}(t) = \dot{\vec{v}}(t) = \begin{pmatrix} \dot{v}_x(t) \\ \dot{v}_y(t) \end{pmatrix} = \begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{pmatrix} = \begin{pmatrix} a_x(t) \\ a_y(t) \end{pmatrix}$$

module

$$|\vec{v}(t)| = \left| \begin{matrix} v_x(t) \\ v_y(t) \end{matrix} \right| = \sqrt{(v_x(t))^2 + (v_y(t))^2}$$

$$\tan(\alpha) = \frac{V_y}{V_x}$$

Projectile motion with $\vec{g}$ constant

$$\vec{a}(t) = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

$$\vec{v}(t) = \begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} v_{x,0} \\ v_{y,0} - gt \end{pmatrix} = \vec{v}_0 + \vec{a}t \;, \quad \vec{v}_0 = \begin{pmatrix} v_{x,0} \\ v_{y,0} \end{pmatrix}$$

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} v_{x,0} \\ v_{y,0} \end{pmatrix} \cdot t + \frac{1}{2}\begin{pmatrix} 0 \\ g \end{pmatrix} \cdot t^2$$

$$v_{x,0} = |\vec{v}_0| \cdot \cos\alpha = v_0 \cdot \cos\alpha$$

$$v_{y,0} = v_0 \cdot \sin\alpha$$

Forces

1st axiom: Law of inertia

$$\vec{F}_{net} = 0 \iff \vec{a} = 0 \iff \vec{v} = \text{constant}$$

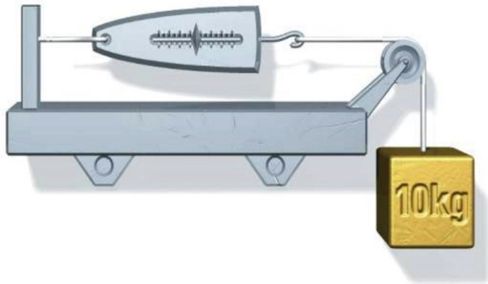
2st axiom: Equation of motion

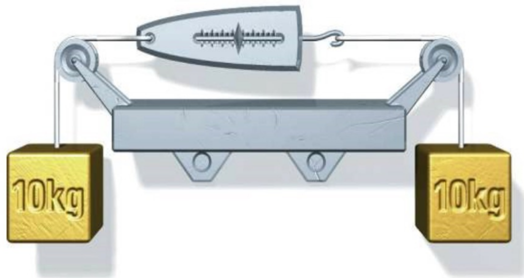
$$\vec{F}_{net} = m \cdot \vec{a} \iff \begin{cases} F_x = ma_x \\ F_y = ma_y \\ F_z = ma_z \end{cases}$$

3rd axiom: Law of mutual interaction

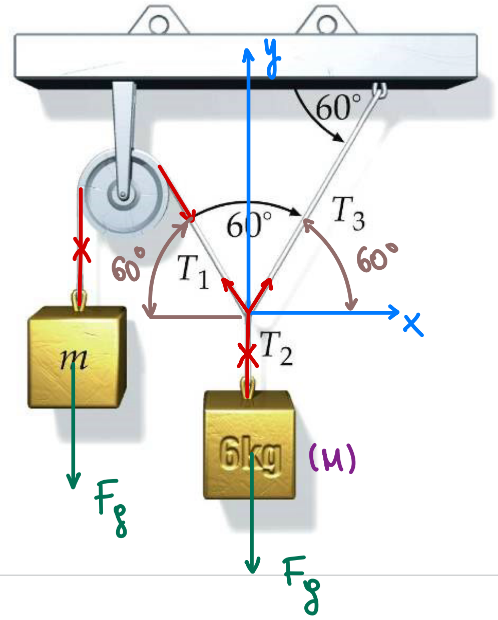
$$\vec{F}_{A \to B} = -\vec{F}_{B \to A}$$

Forces distribution





Tension



$$\Sigma T = 0$$

$$T_2 = F_g = Mg \Rightarrow \vec{T}_2 = \begin{pmatrix} 0 \\ -Mg \end{pmatrix}$$

$$\vec{T}_3 = \begin{pmatrix} T_3 \cos 60° \\ T_3 \sin 60° \end{pmatrix}$$

$$\vec{T}_1 = \begin{pmatrix} -T_1 \cos 60° \\ T_1 \sin 60° \end{pmatrix}$$

$$\vec{T}_1 + \vec{T}_2 + \vec{T}_3 = \begin{pmatrix} -mg\cos 60° \\ mg \sin 60° \end{pmatrix} + \begin{pmatrix} 0 \\ -Mg \end{pmatrix} + \begin{pmatrix} T_3 \cos 60° \\ T_3 \sin 60° \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x: \; -mg\cos 60° + 0 + T_3 \cos 60° = 0$$

$$T_3 = \frac{mg\cos 60°}{\cos 60°} = mg$$

$$y: \; mg\sin 60° - Mg + T_3 \sin 60° = 0$$

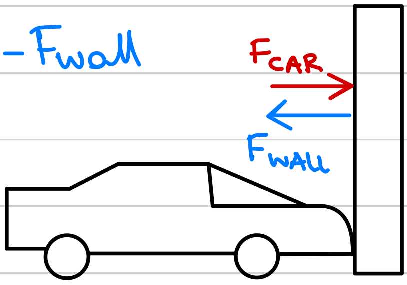
$$mg \sin 60° - Mg + mg\sin 60° = 0$$

$$2m\sin 60° = M \Rightarrow m = \frac{1}{2}\frac{M}{\sin 60°} = 3{,}46 \text{ kg}$$

$$T_1 = T_2 = mg = 3{,}46\,\text{kg} \cdot 9{,}81\,\text{m/s}^2 = 33{,}94\,\text{N}$$

3rd law example

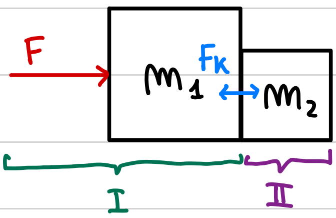
$F_{car} = -F_{wall}$



$m = 1{,}5t$
$v = 60\,\text{km/h}$

Mutual forces

$m_1 > m_2$



$$\text{I}: m_1 \cdot a = F - F_K$$

$$\text{II}: m_2 \cdot a = F_K$$

$$\text{I}+\text{II}: m_1 a + m_2 a = F - F_K + F_K$$

$$a(m_1 + m_2) = F$$

Spring forces, Hooke's law

$F_K = k \cdot x$; $\vec{F}_H = -k\vec{x}$

Contact force with support or wall

Static friction force

$$F_A = F_{SF} < \mu_{SF} \cdot F_N$$

Kinetic friction force

$$F_{KF} = \mu_{KF} \cdot F_N$$

Note: $\mu_{KF} < \mu_{SF}$

Dry friction (μ_{kF}, μ_{SF})

$$\vec{F}_R = \mu_R \cdot |\vec{F}_N| \cdot \frac{-\vec{v}}{|\vec{v}|}$$

Small velocities, viscous forces

$$\vec{F}_{LW,l} = b \cdot (-\vec{v})$$

Large velocities (e.g. air resistance)

$$\vec{F}_{LW,s} = c \cdot \vec{v}^2 \left(-\frac{\vec{v}}{|\vec{v}|}\right.$$

Rotational motion

position: $\vec{r}(t) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \cdot \cos\theta(t) \\ r \cdot \sin\theta(t) \end{pmatrix} = \begin{pmatrix} r \cdot \cos(\omega t) \\ r \cdot \cos(\omega t) \end{pmatrix}$

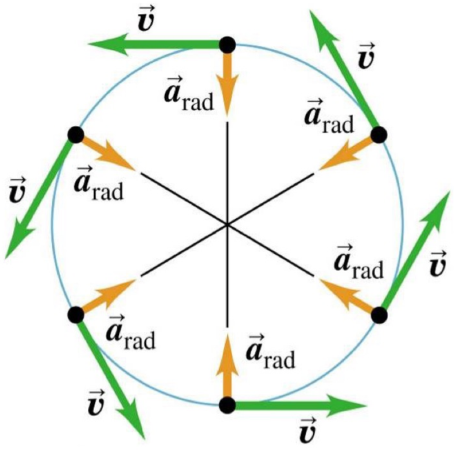
velocity: $\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \begin{pmatrix} -r\omega \sin(\omega t) \\ r\omega \cos(\omega t) \end{pmatrix} = r\omega \begin{pmatrix} -\sin(\omega t) \\ \cos(\omega t) \end{pmatrix}$

magnitude: $|\vec{v}(t)| = r\omega \sqrt{\sin(\omega t)^2 + \cos(\omega t)^2} = r\omega \cdot 1 = r\omega$

acceleration: $\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = r\omega^2 \begin{pmatrix} -\cos(\omega t) \\ -\sin(\omega t) \end{pmatrix}$

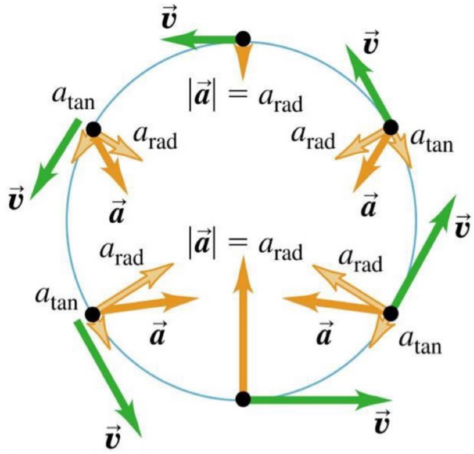
magnitude: $|\vec{a}(t)| = r\omega^2 = a_{cp} = \frac{v^2}{r}$

$$\omega = \frac{\theta(t)}{t} = \frac{2\pi}{T} = 2\pi f$$



Uniform circular motion

$$a_{radial} = a_{CP} = \frac{v^2}{r} = \omega^2 r$$



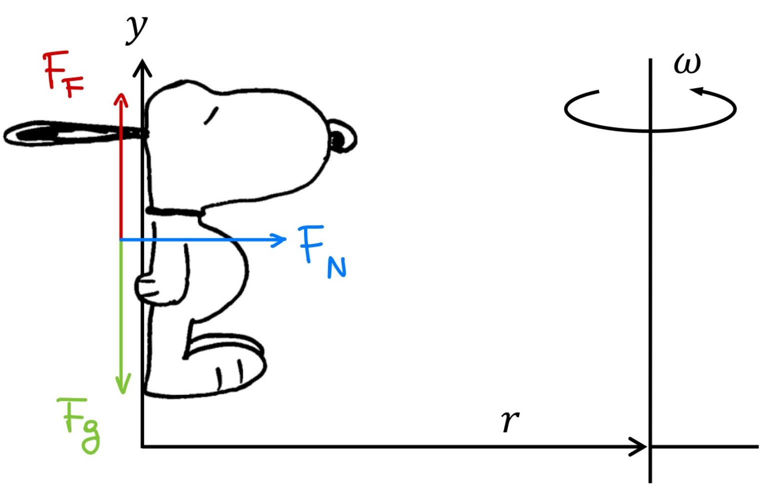
Non-uniform circular motion

$$a_{radial} = a_{CP} = \frac{v^2}{r} = \omega^2 r \qquad a_{tangent} = \frac{d|v|}{dt}$$

Forces in circular motion

$$F_{rad} = F_{CP} = m \cdot a_{CP} = m\frac{v^2}{r} = m\omega^2 r = m(2\pi f)^2 r = m\left(\frac{2\pi}{T}\right)^2 r$$

Example:



Work

$$W = F_{\parallel} d = F\cos\theta \cdot d = \vec{F} \cdot \vec{d}$$

Work with variable force

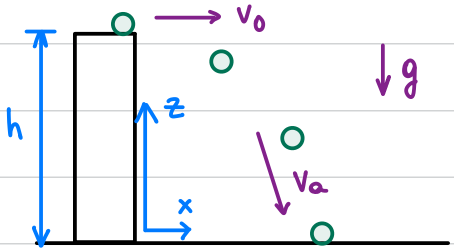
$$W = \sum_i F_{\parallel}(x_i) \cdot dx_i = \int_A^B F_x \cdot dx$$

Lifting work and potential energy

$$W_{PUSH} = F_x \cdot \Delta x = mg\sin\theta \cdot \Delta x = mg\Delta h = \Delta E_{POT}$$

Work stored as energy

Motion and energy



$$\vec{F} = m\vec{a} = \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} \longrightarrow \quad x(t) = x(0) + \int_0^t v(t')\,dt'$$

$$v_x(t') = v_x(0) + \int_0^{t'} v(t'')\,dt''$$

$$\vec{v}(t) = \begin{pmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \end{pmatrix} = \begin{pmatrix} v_0 \\ 0 \\ -gt \end{pmatrix}$$

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} v_0 \cdot t \\ 0 \\ h - \frac{1}{2}gt^2 \end{pmatrix}$$

$$z(t) = 0 \longrightarrow t_a = \sqrt{\frac{2h}{g}} \longrightarrow v_a = \sqrt{v_0^2 + 2gh}$$

when all $E_p \to E_k$

$$v = \sqrt{2gh}$$

Power

$$P = \frac{dW}{dt} = \frac{\vec{F}\,d\vec{s}}{dt} = \vec{F} \cdot \vec{v}$$

Average power

$$\langle P \rangle_{dt} = \frac{\int_{t_1}^{t_2} P(t)\,dt}{t_2 - t_1} = \frac{W}{\Delta t}$$

Momentum and force

Newton: $\vec{F} = m \cdot \vec{a}$, $m = \text{const.}$

$= m \cdot d\vec{v}/dt$

Original way: $\vec{F} = \dfrac{d}{dt}(m \cdot \vec{v}) = \dfrac{d\vec{p}}{dt}$

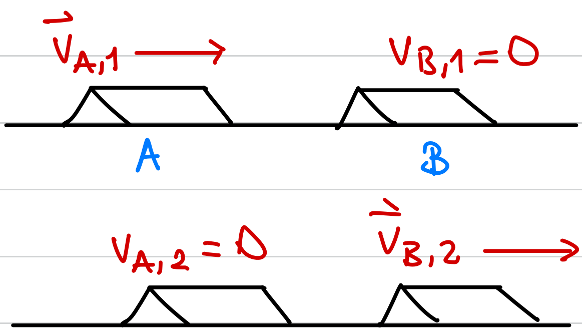
Momentum: $\vec{p} = m \cdot \vec{v}$

Conservation of linear momentum

$$\vec{F}_{net} = \sum_k \vec{F}_k = 0 \longleftrightarrow \vec{P}_{tot} = \text{constant}$$

Momentum conservation for elastic collisions

$$\vec{F}_x = 0 = \frac{\Delta \vec{p}_{TOT}}{\Delta t} \; ; \; \vec{p}_{TOT} = \text{constant}$$



1 $\Rightarrow P_{TOT,1} = P_{A,1} + P_{B,1}$

$= m \cdot V_{A,1} + 0$

2 $\Rightarrow P_{TOT,2} = P_{A,2} + P_{B,2}$

$= 0 + m \cdot V_{B,2}$

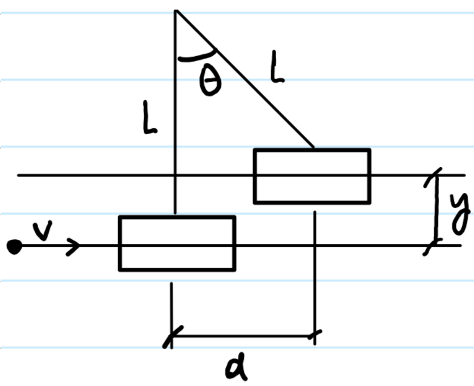
$P_{TOT_1} = P_{TOT,2} \Leftrightarrow m \cdot V_{A,1} = m \cdot V_{B,2}$

$\Rightarrow m_1 = m_2 \Rightarrow V_{A,1} = V_{B,2}$

Completely inelastic collision

$$m_A \vec{v}_{A,1} + m_B \vec{v}_{B,1} = m_A \vec{v}_{A,2} + m_B \vec{v}_{B,2}$$

$$\vec{v}_1 (m_A + m_B) = \vec{v}_2 (m_A + m_B)$$



$$L - y = L \cdot \cos(\theta) \Rightarrow y = L - L \cdot \cos\theta$$

$$\frac{d}{L} = \sin\theta \quad ; \quad \theta = \sin^{-1}\left(\frac{d}{L}\right)$$

Laws of conservation

Interested in initial / final state

- conservation of energy:
  - system's boundaries
  - (non-) conservative forces

Center of mass (cm)

$$\bar{x}, \bar{y}, \bar{z} = \frac{\sum \tilde{x}, \tilde{y}, \tilde{z} A}{\sum A}$$

Area can be changed with distance, weight, velocity, ...

It depends on the situation.

Fluids

Pressure in fluids

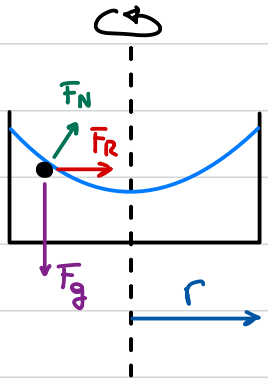
$$p = \frac{F_\perp}{A} = \frac{dF_\perp}{dA}$$

$$p(y) = p_0 - \rho g \Delta h$$

Pascal's principle

$$\frac{F_1}{A_1} = p_1 = p_2 = \frac{F_2}{A_2}$$

Accelerated fluids



$$F_{RES,r} = m\omega^2 r \quad (\text{radial})$$

$$= F_N \sin\beta$$

$$F_{RES,y} = -F_g + F_N \cos\beta \quad (\text{vertical})$$

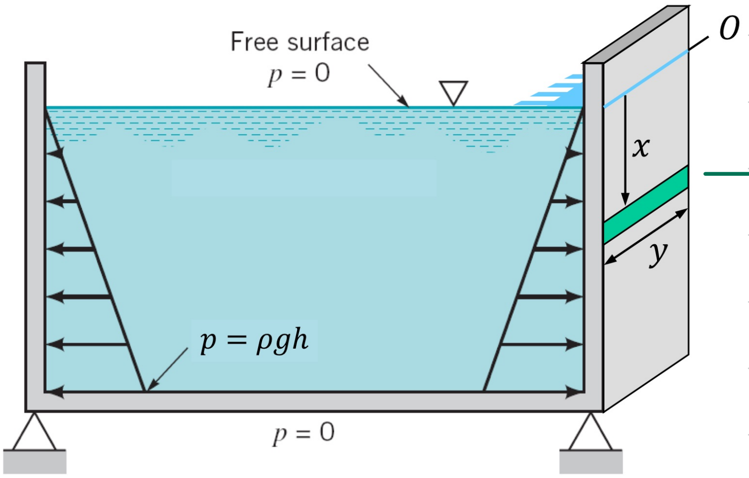
$$\tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{\omega^2 r}{g}$$

$$\updownarrow$$

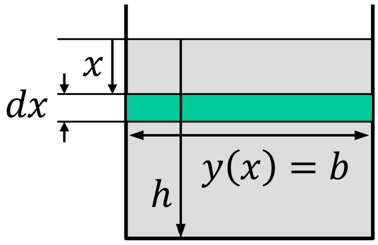
$$\frac{h(r)}{dr} \Rightarrow h(r) = \int \frac{\omega^2 r}{g}\, dr$$

$$= \frac{\omega^2}{2g} \cdot r^2 \quad \}\ \text{parabola}$$

Pressure on walls



force on a small area



$$P_i = \rho_w g x_i$$

$$F_i = P_i \cdot A = P_i \cdot dx_i \cdot b$$

$$F_{TOT} = \sum_i F_i = \sum_i P_i \cdot dx_i \cdot b$$

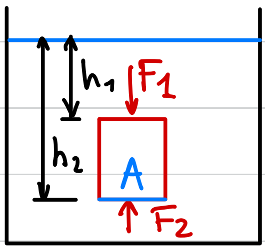
Continuous problem:

$$F_{TOT} = \int_0^h p(x) \cdot b \cdot dx = \int_0^h \rho g x \cdot b \cdot dx$$

Buoyancy, Archimedes' principle

$$F_A = \rho_{Fluid} \cdot g \cdot V_{Body, immersed}$$

Derivation 1:



$$F_A = F_2 - F_1$$
$$= P_2 A - P_1 A$$
$$= \rho_{Fl} g (h_2 - h_1) A = \rho_{Fl} g V_B$$

Continuity equation

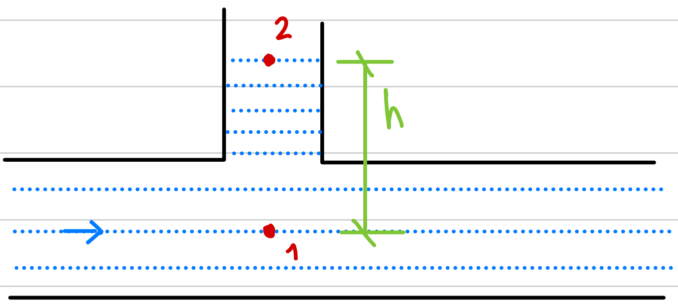
$$I_{vol} = \frac{dV}{dt} = A_1 v_1 = A_2 v_2$$

$$dV_1 = A_1 v_1 dt = A_2 v_2 dt = dV_2$$

Bernoulli's equation

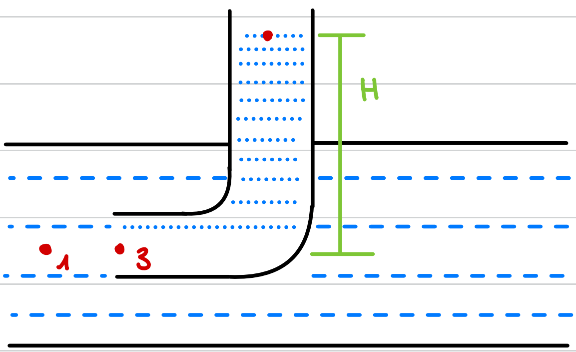
$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2 = \text{const.}$$

Static pressure measurement



$$P_1 = P_{atm} + \rho g h$$

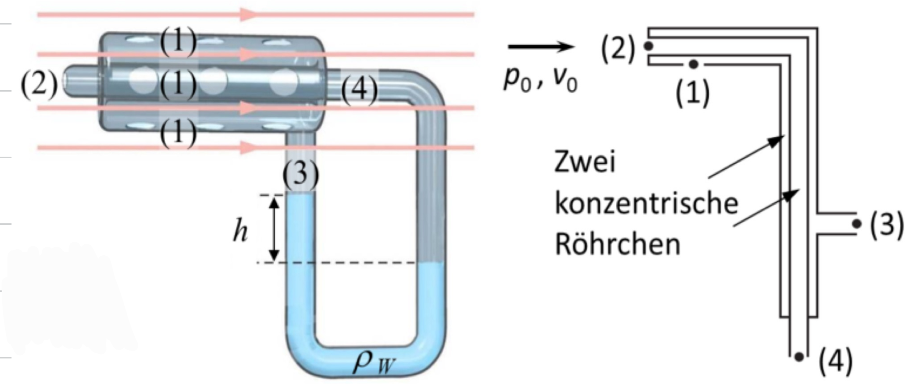
Pitot tube



$$P_1 + \cancel{\rho g \cdot 0} + \frac{1}{2}\rho v_1^2 = P_3 + \cancel{\rho g \cdot 0} + \cancel{\frac{1}{2}\rho \cdot 0}$$

$$\boxed{P_3 = P_1 + \frac{1}{2}\rho v_1^2}$$

Speed measurement with a Pitot tube



$$p_1 = p_3 = p_0$$

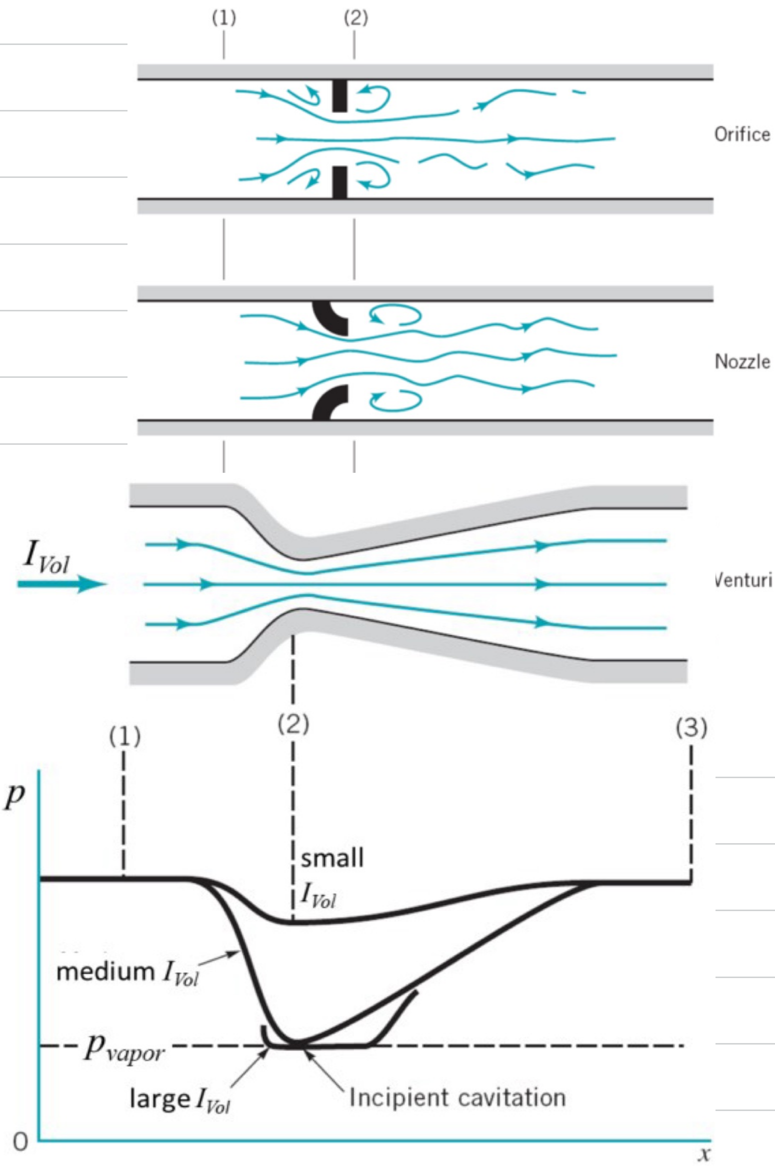
$$p_2 = p_4 = p_0 + \frac{1}{2}\rho v_0^2$$

$$p_4 - p_3 = \rho_W g h = p_2 - p_1 = \frac{1}{2}\rho v_0^2$$

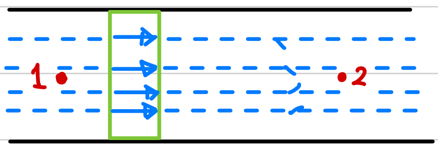
$$\boxed{v_0 = \sqrt{\frac{2(\rho_W g h)}{\rho}}}$$

Cavitation

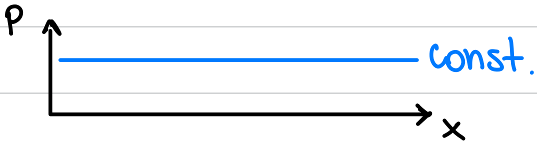
$$p_1 - p_2 = \frac{(I_{vol})^2}{2A_2^2}\rho\left(1-\left(\frac{A_2}{A_1}\right)^2\right)$$



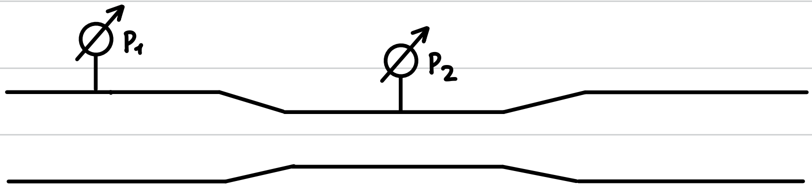
Ideal, frictionless fluids



$$\Delta p = p_2 - p_1$$



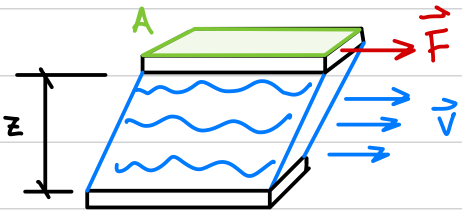
Venturi effect



$$p_1 + \frac{1}{2}\rho v_1^2 = \text{const.} = p_2 + \frac{1}{2}\rho v_2^2$$

$$p_1 - p_2 = \frac{1}{2}\left(I_{vol}\right)^2 \rho \left(\frac{1}{A_2^2} - \frac{1}{A_1^2}\right)$$

Viscosity



$$F \sim \frac{A \cdot v}{z}$$

$$\boxed{\frac{F}{A} = \tau = \eta \frac{dv}{dz}}$$

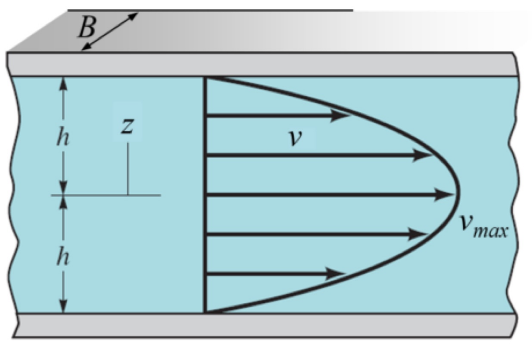
τ = Shear stress

η = dynamic viscosity

$$\tau \propto \frac{dv}{dz}$$

Velocity profile v(z) between parallel plates

$$v(z) = v_{max}\left(1 - \left(\frac{z}{h}\right)^2\right) = \frac{3}{2}\bar{v}\left(1 - \left(\frac{z}{h}\right)^2\right)$$



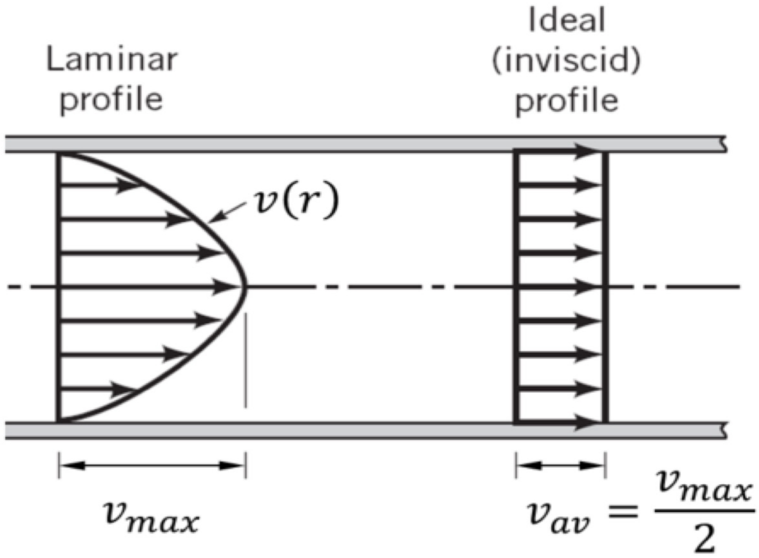
A perfect parabolic profile $v(z)$ is only formed if h is not too large.

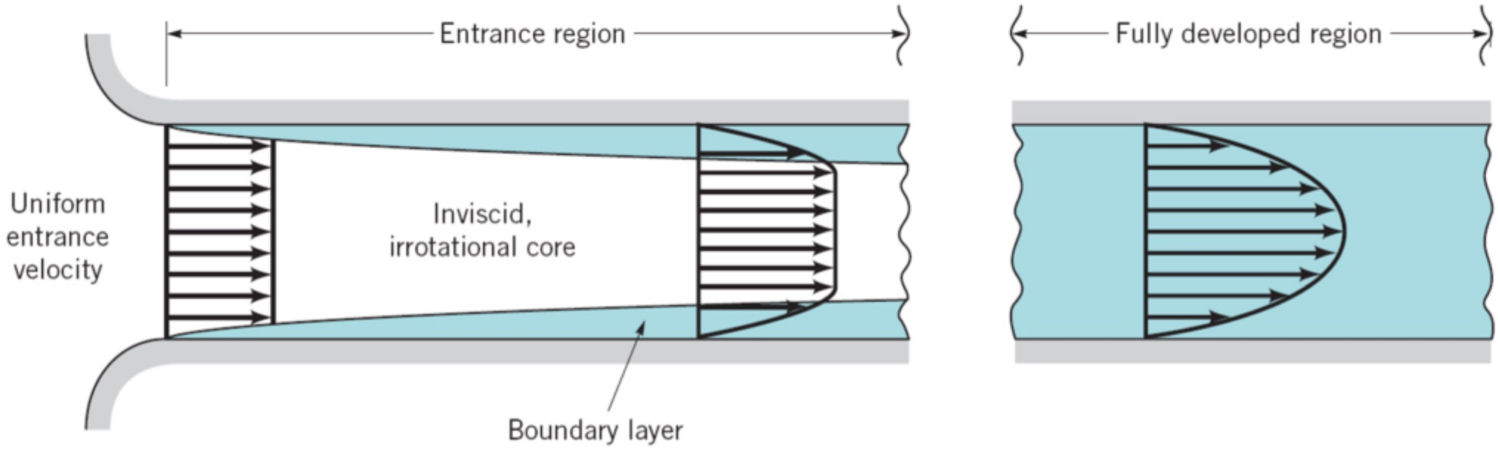
Velocity profile v(r) in a circular pipe

$$v(r) = \frac{\Delta p R^2}{4L\eta}\left(1-\left(\frac{r}{R}\right)^2\right)$$

$$= v_{max}\left(1-\left(\frac{r}{R}\right)^2\right)$$

$$= 2\bar{v}\left(1-\left(\frac{r}{R}\right)^2\right)$$

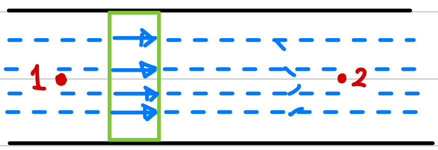


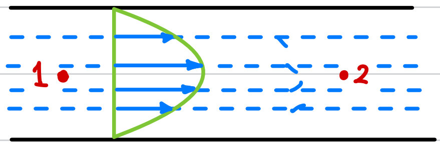


Pressure in tubes

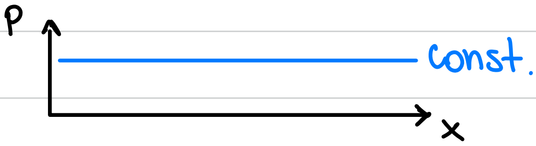
Ideal, frictionless fluids

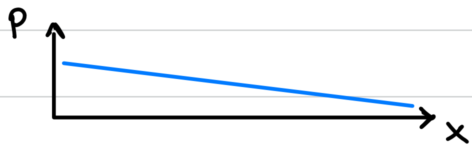
Real fluids



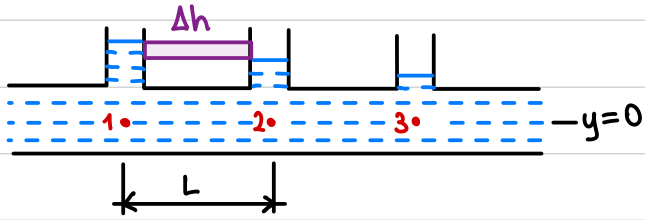


$$\Delta p = p_2 - p_1$$





Piezo tube



$$P_1 = P_2 + \underbrace{\rho g \Delta h}_{= \Delta p} \quad ; \quad \Delta p = \frac{I_{vol} \cdot 8\eta \cdot L}{\pi R^4}$$

Bernoulli: $p + \rho g y + \frac{1}{2}\rho v^2 + \Delta p = \text{constant}$

Hagen – Poiseuille

I_{vol} = volumetric flow rate ($\dot{V}$)

$I_{vol} = A \cdot v_{avg}$

$$I_{vol} = \frac{\pi \cdot R^4}{8L \cdot \eta} \Delta p$$

$$\Rightarrow \Delta p = \frac{128 L \cdot \eta \cdot I_{vol}}{\pi \cdot D^4}$$