Technical Mechanics

2. Semester
HSLU

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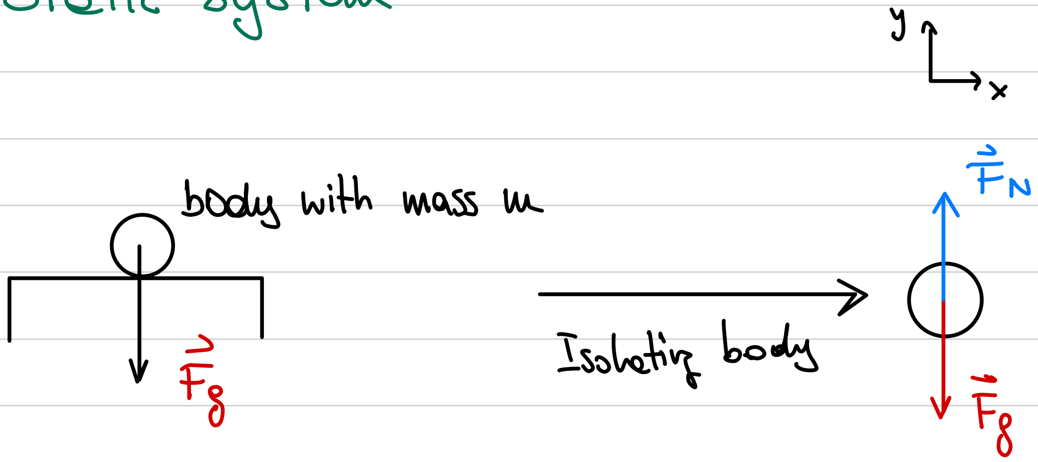
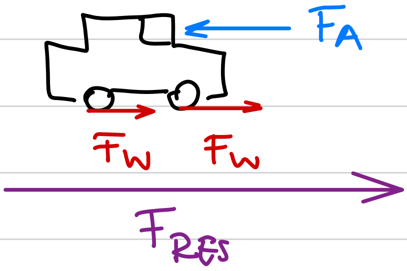
Static system



$$\Sigma F_y = F_N - F_g = 0$$

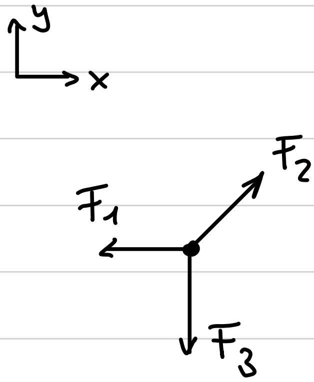
$$\Sigma F_x = 0$$

Dynamic system



$$a = \frac{F_{RES}}{m}$$

Force directions and resultants



$$\Sigma F_x = -F_1 + F_2 \cos(\alpha)$$

$$\Sigma F_y = -F_3 + F_2 \sin(\alpha)$$

Let's assume $\alpha = 45°$ and $F_2 = 100\,N$:

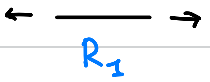
$F_1 = F_2 \cos 45° = 70{,}7\,N$

$F_3 = F_2 \sin 45° = 70{,}7\,N$

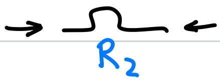
Ropes

Ropes only can take tensile forces and NOT compressive forces

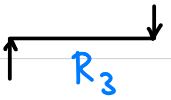
Pull ✓



Push ✗

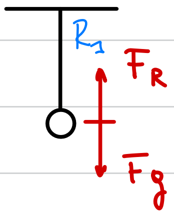


Bend ✗



• Isolated ropes

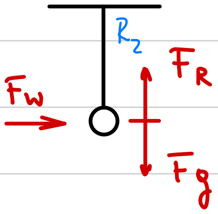
STATIC



$\Sigma F_y = 0$

$\Sigma F_x = 0$

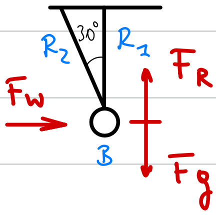
DYNAMIC (with wind)



$\Sigma F_y = 0$

$\Sigma F_x = F_w$ } For make it static, we have to add more ropes

STATIC from a D. state



let's assume: $F_w = 50\ N$

$m_B = 200\ kg$

$\Sigma F_x = F_w - F_{R_1} \cdot \cos(-30°)$

$\Sigma F_x = 50N - F_{R_2} \sin 30° = 0$

$\Sigma F_y = 0$

$\Sigma F_y = F_{R_1} - F_g + F_{R_2} \cos 30° =$

$F_{R_2} = 50N / \sin 30° = 100\ N$

$F_y = F_{R_1} - F_g + F_{R_2} \cos 30°$

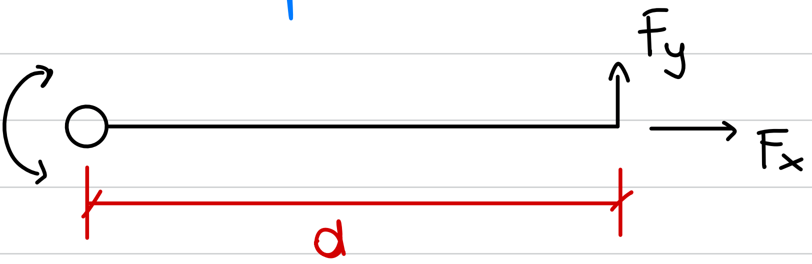
$F_g = mg = 200kg \cdot 9{,}81\ m/s^2 = 1962\ N$

$F_{R_1} = F_g - F_{R_2} \cos(30°) = 1962N - 86{,}6\ N = 1875{,}4\ N$

$F_y = 0$

Moments and couple

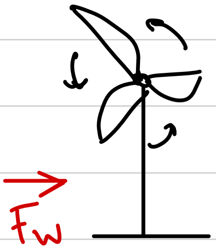
Couple:

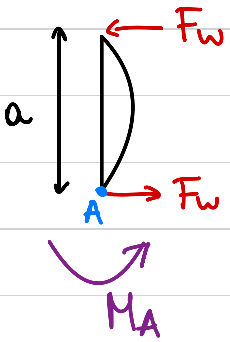


A couple is created by a force applied at a distance

$$\boxed{\begin{array}{l} M_z = F_x d_x \\ M_z = F_y d_y \end{array}}$$

$\Downarrow$ Example



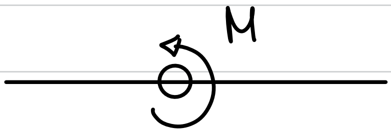


$\Sigma F_A = 0$

$$M_A = F_w \cdot d \rightarrow M = \begin{cases} F_x \cdot d_y \\ F_y \cdot d_x \end{cases}$$

$M [Nm]$

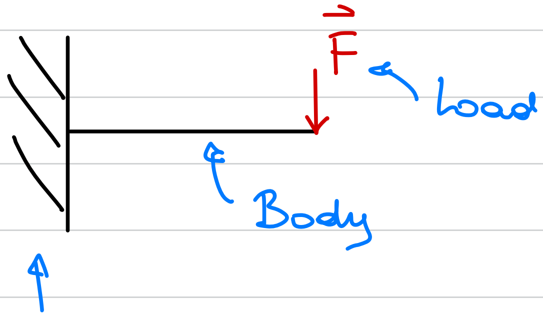
Moments:



A moment is created by an engine and acts at one single point

Free body diagram (FBD)

For each mechanical problem, drawing a FBD is needed!



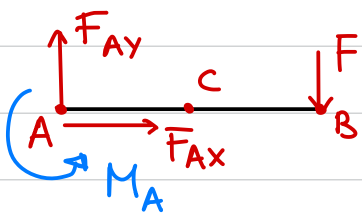
(1)

Boundary ← We need to replace the boundary condition by forces and moments

Boundary conditions can be created by:

- touching bodies
- hinges and fixations
- environmental forces (pressure, gravity)

In (1), we isolate the body:



$$\Sigma F_x = 0 = A_x$$

$$\Sigma F_y = 0 = -F + A_y$$

$$\Sigma M_A = -F \cdot dx + M_A$$

$$\Sigma M_B = M_A - A_y \cdot dx = 0$$

$$\Sigma M_C = M_A - F \cdot \frac{1}{2} dx - A_y \cdot \frac{1}{2} dx$$

$$M_A - \left(\frac{A_x \cdot dx - B_x \cdot dx}{2} \right)$$

Supports

Every blocked degree of freedom (DOF) needs to be replaced by a force or a moment

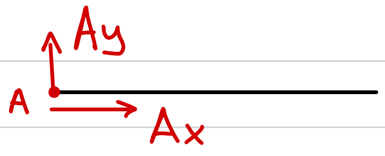
- Rotation blocked → M
- Translation blocked → F

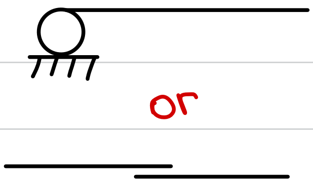
In 2Ds systems there are 3 DOF for each point:

1) Translation in x
2) Translation in y
3) Rotation around z

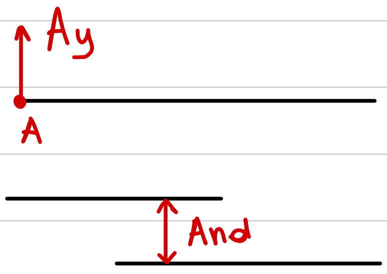
Types of supports

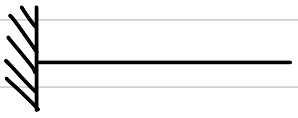
Hinges fix x, y and allow rotation



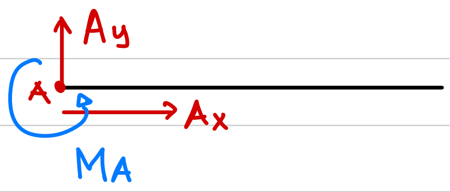


Rollers or two horizontal surfaces fix y and allow rotation and traslation in x

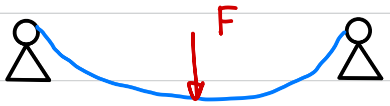




Wall fixtures / Fixed supports fix x,y and rotations

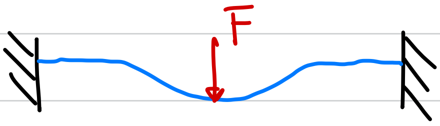


1)



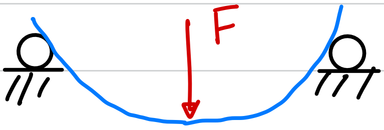
→ Used for shelves

2)



→ Used for shelves

3)



→ used for bending tests, balance boards

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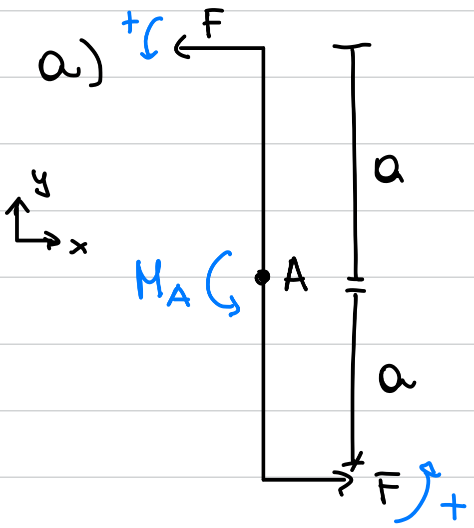
exercises:

a)

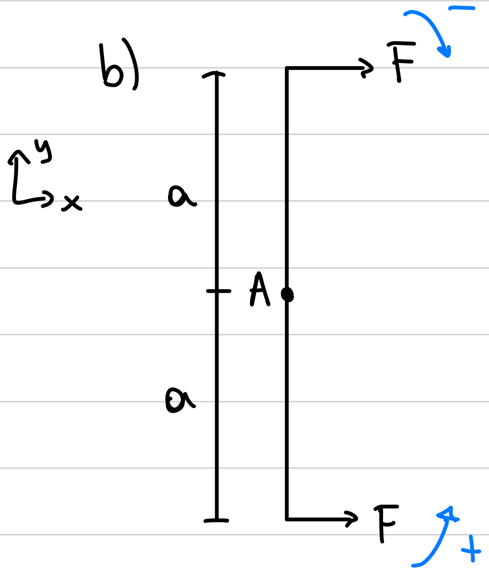


$\Sigma F_x = F - F = 0$

$\Sigma F_y = 0$

$\Sigma M_A = F \cdot a + F \cdot a = 2Fa$

b)

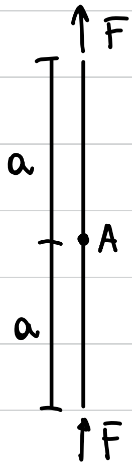


$\Sigma F_x = 2F$

$\Sigma F_y = 0$

$\Sigma M_A = +Fa - Fa = 0$

c)

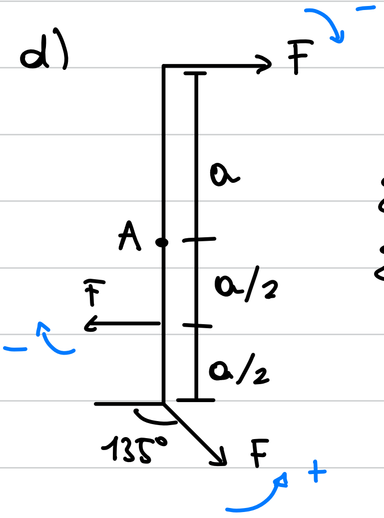


$\Sigma F_y = F + F = 2F$

$\Sigma F_x = 0$

$\Sigma M_A = 0$

d)



$\Sigma F_x = \cancel{F} - \cancel{F} + F\cos 45$

$\Sigma F_y = -F\sin 45$

$\Sigma M_A = -Fa - F \cdot a/2 + F \cdot \cos 45 \cdot a$

Systems with friction and ropes

Ropes and pulleys

Ropes can be only pulled. This means the forces coming from ropes are always in the direction of the rope.

Notice: Pulleys redirect ropes. If the pulley is perfectly frictionless:

$$\boxed{F_1 = F_2}$$

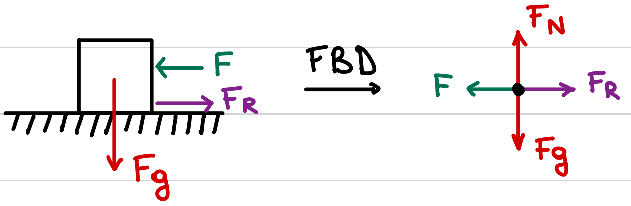
Friction on a surface

The magnitude of the frictional force F_R is determined with two parameters:

- F_N : Normal force from surface ($\perp$)
- Coefficient of friction (static / dynamic) between object and surface:
  - ▷ Static friction: μ_S
  - ▷ Dynamic friction: μ_D

$$\boxed{\begin{aligned} F_R &= \mu_S \cdot F_N \\ F_R &= \mu_D \cdot F_N \end{aligned}}$$

Graphics:



▷ On flat surface: $F_N = F_g$

▷ If the body doesn't move: μ_S

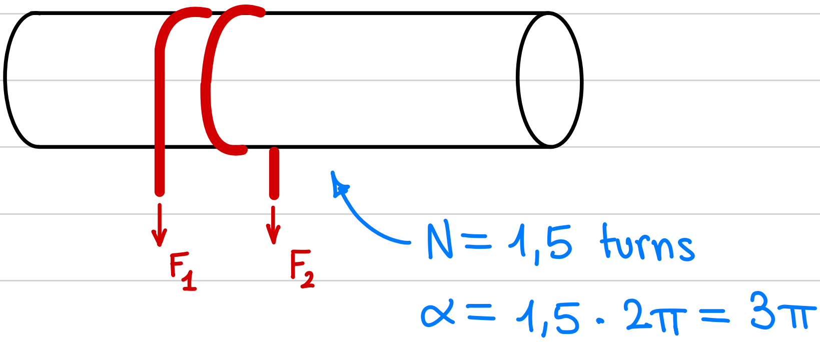
▷ If the body moves: μ_D

Winding rope with friction

The forces on each end of the rope are not equal.
Force F_1 on one end of the rope is given by:

$$F_1 = e^{\mu_D \cdot \alpha} \cdot F_2$$

where: μ_D = Dynamic friction
α = winding angle [rad]

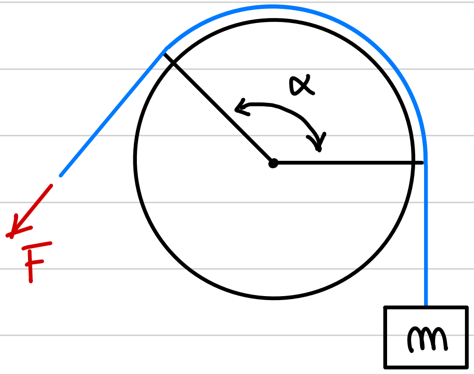


In a pulley with friction we can also apply this formula:

$$F = e^{\mu_D \cdot \alpha} \cdot F_g$$

Friction force in the pulley:

$$F_{R,P} = (e^{\mu_D \cdot \alpha} - 1) \cdot F_1$$



Note on units:

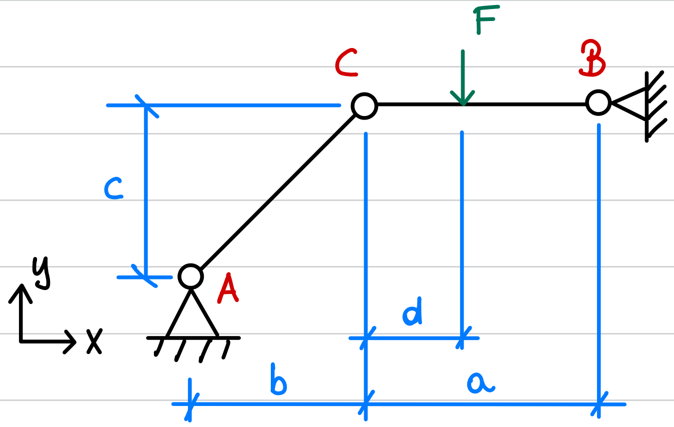
Force: $[N] = [kg] \cdot [m \cdot s^{-2}]$

Angle:

$$[rad] = \frac{180 \cdot [°]}{\pi}$$

Multi - body systems

Two bodies can have several FBD's :



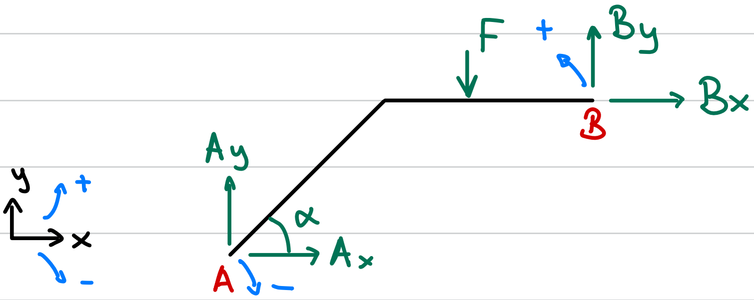
Example

Let :

$F = 2000\ N$, $a = 7m$,

$b = 2m$, $c = 6m$, $d = 3m$.

Step 1 : Set up the FBD of the entire system:



Step 2 : Equilibrium equations for $F\begin{pmatrix} A_x, B_x \\ A_y, B_y \end{pmatrix}$ seen from point B :

$$\Sigma F_x = F(A_x) + F(B_x) = 0$$

$$\Sigma F_y = F(A_y) + F(B_y) - F$$

where:

$F(A_x) = F_A \cdot \cos\alpha$

$F(A_y) = F_A \cdot \sin\alpha$

$$\Sigma M_B = F(A_x) \cdot c - F(A_y)(a+b) + F(a-d) = 0$$

Step 3 : Magnitude / Direction :

$$\tan\alpha = \frac{c}{b}$$

Step 4: Final calculations:

$$\Sigma M_B = 0$$

$$0 = F_A \cos\alpha \cdot 6 - F_A \sin\alpha\,(7+2) + 2000\,(7-3)$$

$$F_A = \underline{1204{,}7\ N}$$

$$F(A_x) = F_A \cos\alpha = \underline{381\ N}$$

$$F(A_y) = F_A \sin\alpha = \underline{1142{,}9\ N}$$

$$\Sigma F_x = 0$$

$$F(A_x) + F(B_x) = 0$$

$$F(B_x) = \underline{-381\ N}$$

$$\Sigma F_y = 0$$

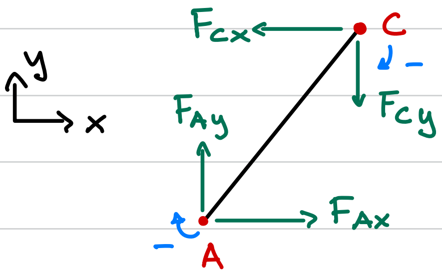
$$F(A_y) + F(B_y) - F = 0$$

$$F(B_y) = F - F(A_y)$$

$$F(B_y) = 2000\ N - 1142{,}9\ N = \underline{857{,}1\ N}$$

Step 5: Forces in the joint

Body 1:



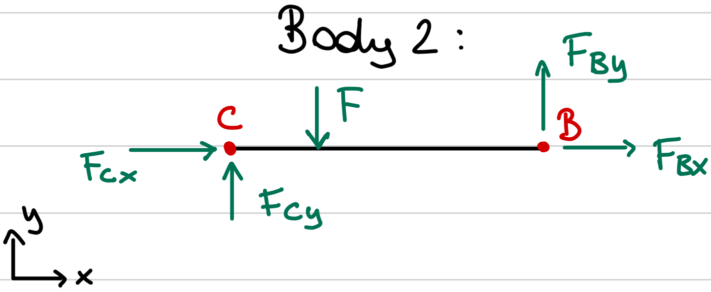
$$\Sigma F_x = F_{Ax} - F_{cx} = 0$$

$$F_{cx} = F_{Ax} = \underline{381\ N}$$

$$\Sigma F_y = F_{Ay} - F_{cy} = 0$$

$$F_{cy} = F_{Ay} = \underline{1141{,}9\ N}$$

Body 2:



$$\Sigma F_x = F_{Bx} + F_{cx} = 0$$

$$F_{cx} = -F_{Bx} = \underline{381\ N}$$

$$\Sigma F_y = F_{By} + F_{cy} - F$$

$$F_{cy} = F - F_{By} = \underline{1141{,}9\ N}$$

Constrains and Statical Determinancy

Statically determinate:

No. of eq. = No. of unknowns

Support forces = DOF

Statically indeterminate:

No. of eq. < No. of unknowns

support forces < DOF

Statically overdeterminated

No. of eq. > No. of unknowns

support forces > DOF

Examples:

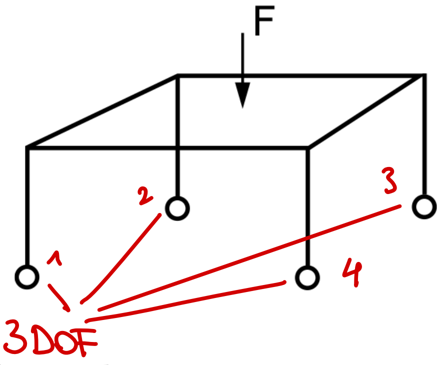
a) Table with 4 legs. All 4 legs on rollers on a flat floor.

3 DOF / leg

4 legs

⇓

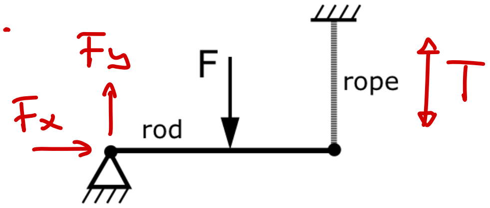
Statically indet.



b) A rod supported by a hinge and a rope

$F_x, F_y, T = 3$ unk.

2D → 3 DOF



3DOF = 3 unk

⇓

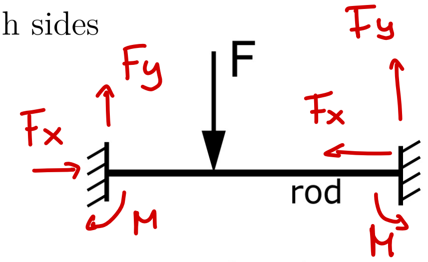
Statically det.

c) A rod fixed on both sides

6 unk, 3DOF

⇓

Statically ind.

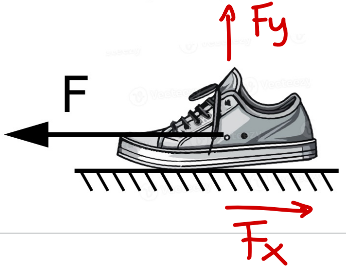


d) A shoe on the ground without slipping (static)

3DOF, 2 unk.

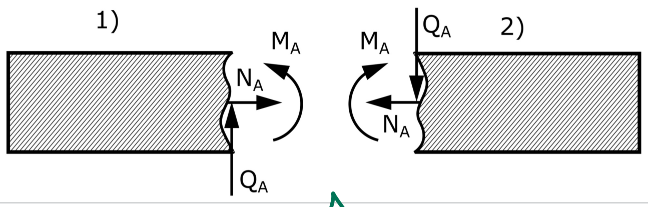
⇓

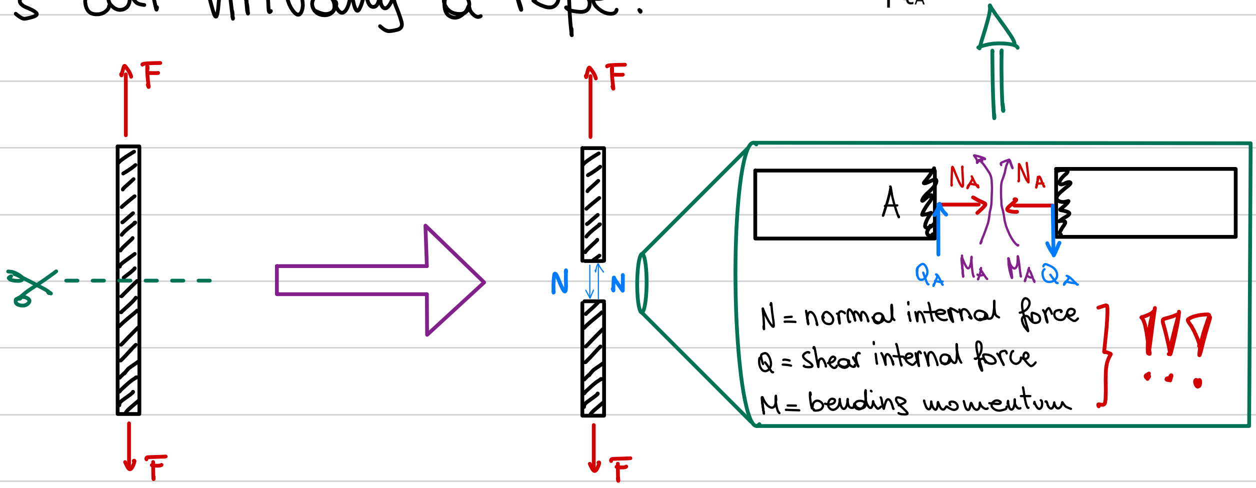
statically overdet



Internal forces

Let's cut virtually a rope:





N = normal internal force
Q = shear internal force
M = bending momentum

!!!

Reactions with less unknowns than equations:

Missing N:

Missing Q:

Missing M:

What do we use for what?

① System FBD and equilibrium: Determining external forces and support reactions

② Body isolation in a multi-body system: Determining interface and reaction forces support reactions

③ Internal forces: Determining stress and evaluate the safety

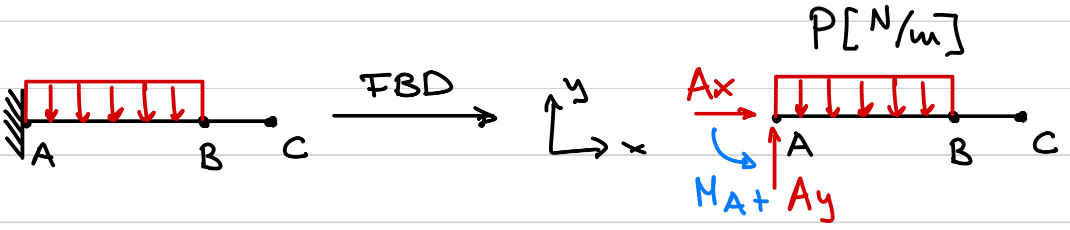
Shear / Moment / Tension diagram

Procedure:

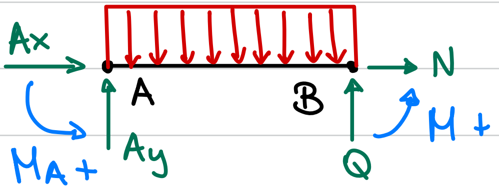
Step 1: FBD diagram, calculate support reactions

Step 2: Calculate internal forces and moment

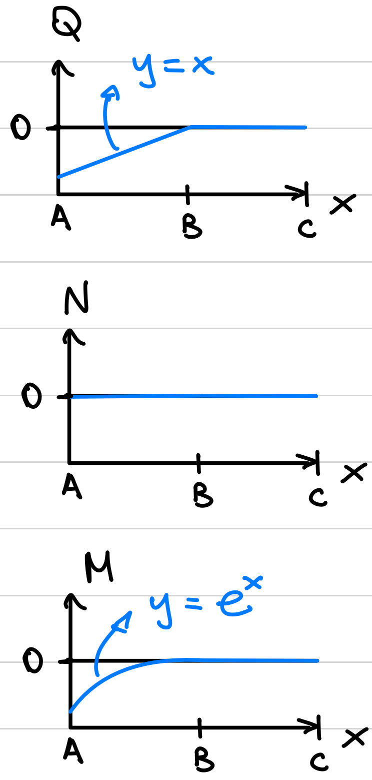
Distributed load



↳ cutting at B:



SMT:



$$M_{DISTR.} = F_{DISTRIBUTED} \cdot l \cdot \frac{1}{2} l \quad [Nm]$$

↓

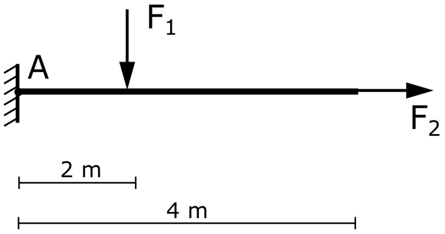
$$\sum F_x = 0 = A_x + N \quad [N]$$

$$\sum F_y = 0 = A_y + Q \quad [N]$$

$$\sum M = 0 = M_A - F_{DISTR} \cdot d \cdot \frac{d}{2} + M + \underline{Q \cdot d} \quad [Nm]$$

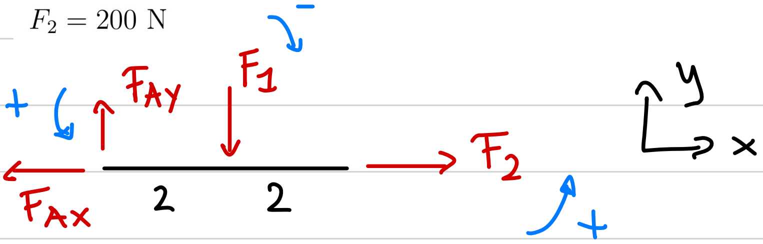
Never forget Q · d !!!

Example



Given:
$F_1 = 100$ N
$F_2 = 200$ N

FBD :

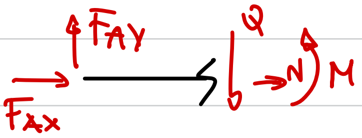


$$\Sigma F_x = -F_{AX} + F_2 = 0 \Rightarrow F_{AX} = F_2 = 200 N$$

$$\Sigma F_y = F_{Ay} - F_1 = 0 \Rightarrow F_{Ay} = F_1 = 100 N$$

$$\Sigma M_A = -F_1 \cdot 2m + F_2 \cdot 4m = 200 N$$

Cut 1: $0m < x < 2m$

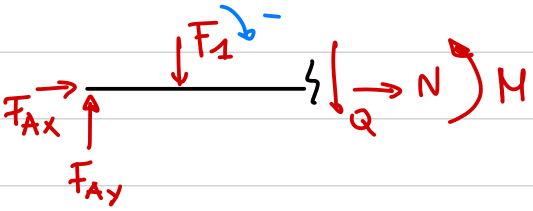


$$N_1 = F_{AX} = 200N$$

$$Q_1 = -F_{AY} = -100 N$$

$$\Sigma M = M_A + Q_1 x$$

Cut 2: $2m < x < 4m$

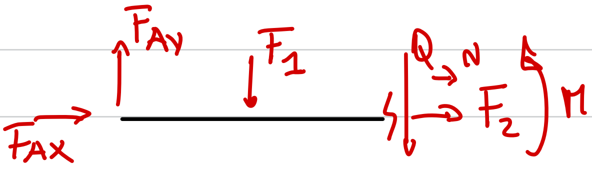


$$N_2 = F_{AX} = 200N$$

$$Q_2 = -F_{Ay} + F_1 = 0N$$

$$\Sigma M = M_A - M_{F_1} + Q_2 x$$

Cut 3:



$N_3 = \ldots$

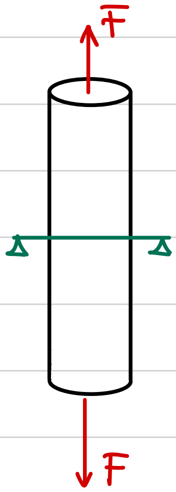
Stress and bending

Stress:

- It is needed to evaluate the safety
- It is calculated differently for each load case
  - Tensile (pure tensile load-stress)
  - Compressive (pure compressive load-stress)
  - Bending (tensile + compressive + shear stress)
  - Shear (pure shear stress)
  - Torsion (pure shear stress)

calculated in the same way

Tensile and compressive stress

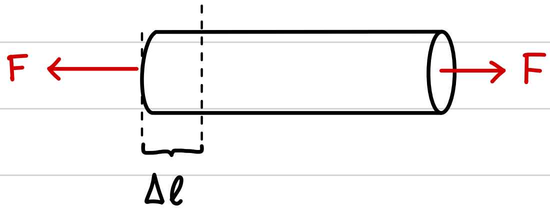


$$\boxed{\sigma_{Tensile} = \frac{F_{int}}{A} \quad \left[MPa = \frac{N}{mm^2}\right]}$$

Stress: internal loads incl. geometry

Strain

strain: internal shape changes



$$\varepsilon_{Tensile} = \frac{\Delta l}{l_0} \quad [-]$$

$$\varepsilon_{Compressive} = \frac{\Delta l}{l_0} \quad [-]$$

$$\gamma_{Shear} = \frac{\Delta S}{\Delta h} \quad [-]$$

$$\boxed{\sigma = E \cdot \varepsilon}$$

E = young's modulus [MPa] or [GPA]

$$\Delta l_{TOT} = \sum_i \Delta l_i \quad ; \quad \varepsilon_{TOT} = \frac{\Delta l_{TOT}}{l_{TOT}}$$

Some young's modulus

$$E_{steel} = 210'000 \text{ MPa} = 210 \text{ GPa}$$

$$E_{Aluminium} = 68'000 \text{ MPa} = 68 \text{ GPa}$$

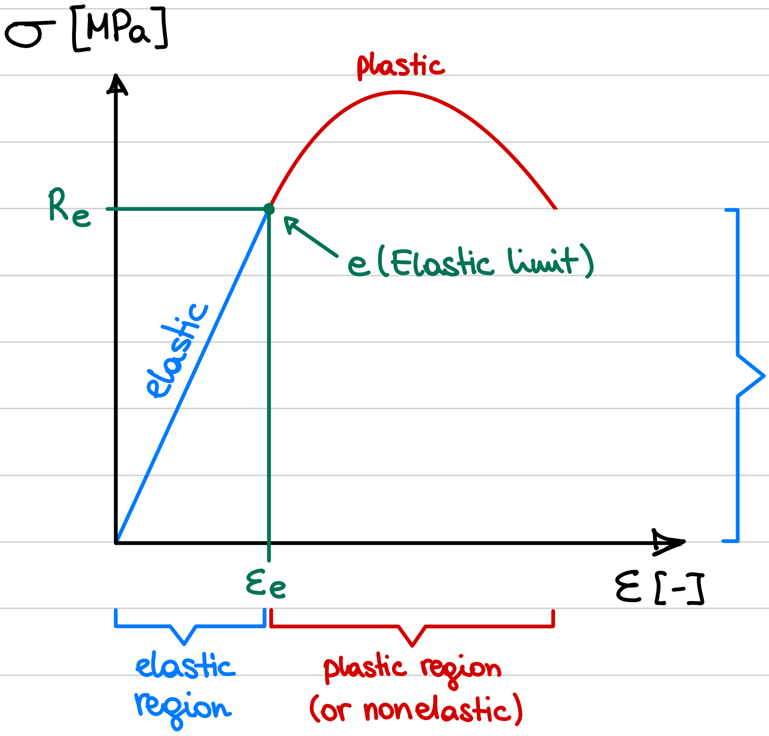
$$E_{PA\ (Polymer)} = 2'100 \text{ MPa} = 2{,}1 \text{ GPa}$$

Safety calculation

$$\left.\begin{aligned} \sigma_{int} &< \sigma_{max,\ admissible} \\ \varepsilon_{int} &< \varepsilon_{max,\ admissible} \end{aligned}\right\} \begin{matrix} \text{material data} \\ + \\ \text{safety factor} \end{matrix}$$

Mechanical properties of materials

Stress-strain diagram



In the elastic zone, we can determine:

- stiffness $\left(\frac{\Delta \varepsilon_e}{\Delta R_e}\right)$
- R_e (maximum stress in the elastic region)
- ε_e (maximum strain in the elastic region)

Note: point e must be completely elastic!

The Young's modulus is now redefined as $E := \frac{\Delta\sigma}{\Delta\varepsilon}$ [MPa]

or, if $(\sigma_0, \varepsilon_0) = (0,0)$, $E = \frac{R_e}{\varepsilon_e}$ [MPa]

Safety factors

Knowing that (ε_e, R_e) is the elastic limit point:

$$\sigma_{max,adm} = \frac{R_e}{S} \quad ; \quad \varepsilon_{max,adm} = \frac{\varepsilon_e}{S}$$

where S is the safety factor defined as

$$S := \frac{\sigma_y}{\sigma_{max,adm}}$$

where:

σ_y = material strength

$\sigma_{max,adm}$ = max allowable stress

$$\sigma_{required} > \sigma \cdot S$$

Typical values for safety factors:

- One load cycle: $S_{static} \approx 1{,}2$ [-]
- Few load cycles $(10^1 \dots 10^4)$: $S_{dyn} \approx 2 \dots 3$ [-]
- Many load cycles $(10^4 \dots 10^6)$: $S_{dyn} \approx 3 \dots 10$ [-]

Load case specific safety factors:

- Bending (pure): $S_z = 1 \dots 1{,}2$ [-]
- Torsion (pure): $S_z = 1 \dots 1{,}2$ [-]
- Torsion and bending: $S_z = 1{,}4$ [-]

$$\left.\right\} \; S_{TOT} = S_z \cdot S_S \cdot S_f \cdot \dots$$

Flow chart for mechanical Stress evaluation:

Start

↓

Evaluate the stress in the structure

↓

use safety factors for shape and geometry (size, notch, ...)

↓

use safety factors for the median stress (pretension)

↓

use safety factors for the environment (corrosion, heat, ...)

↓

use safety factors for the type of load

↓

comparison with material values

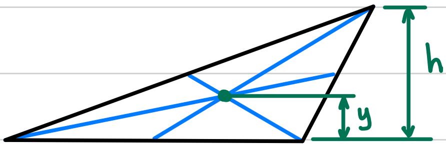
- Tensile stress, static: R_e, ε_e
- Tensile stress, dyn: s-n curves (Wöhler's curve)
- Torsion: τ_{max}, γ_{max}

↓

done

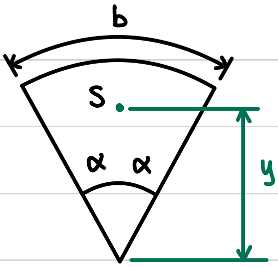
Centroid (center of gravity)

Triangle:



$$\boxed{y = \frac{h}{3}}$$

Circular Sector



$$y = \frac{2r \cdot \sin\alpha \cdot 180}{3\pi\alpha} \; ; \; \alpha [°]$$

Centroid for composed surfaces

$$\bar{x}, \bar{y}, \bar{z} = \frac{\int_V \tilde{x}, \tilde{y}, \tilde{z} \, dv}{\int_V dv}$$, where $\bar{x}, \bar{y}, \bar{z}$ = coordinates of the center of gravity G of the body

$\tilde{x}, \tilde{y}, \tilde{z}$ = coordinates of the center of gravity G of one single entity

$$\bar{x} = \frac{\sum \tilde{x} W}{\sum W} \; ; \; \bar{y} = \frac{\sum \tilde{y} W}{\sum W}$$

where $\sum W$ = total weight

Translating it into centroid:

$$\boxed{\bar{x} = \frac{\sum \tilde{x} A}{\sum A} \; ; \; \bar{y} = \frac{\sum \tilde{y} A}{\sum A}}$$

Moment of inertia

In bending and torsion, the stiffness of the beam is related to the geometry of the cross section.

For bending: moment of inertia (axial)
For torsion: polar moment of inertia

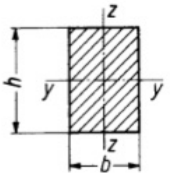
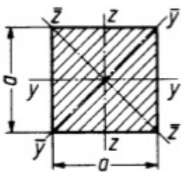
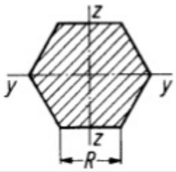
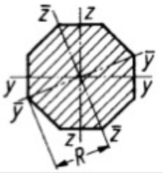
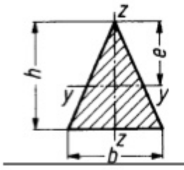
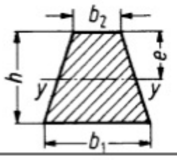
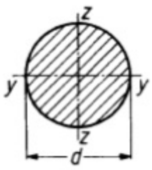
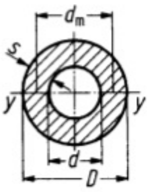
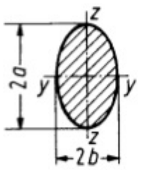
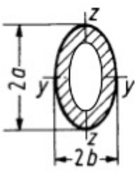
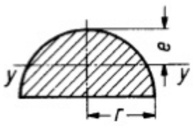
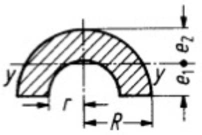
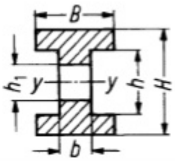
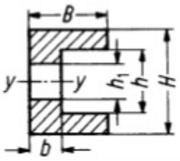
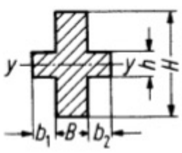
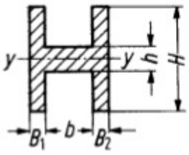
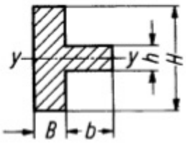
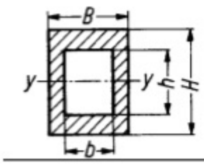
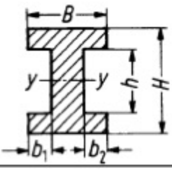
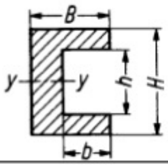
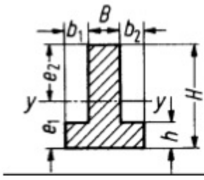
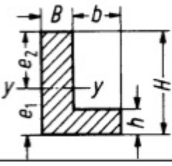
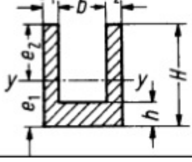
Axial moment of inertia

I_x = Moment of inertia when bending around the x-axis
I_y = Moment of inertia when bending around the y-axis

$I\ [mm^4]$

FORMULA SHEET - MOMENTS OF INERTIA

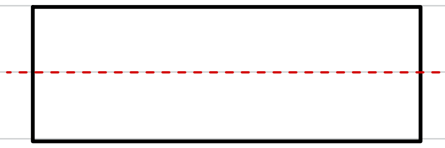
Table. Moment of inertia for different profiles

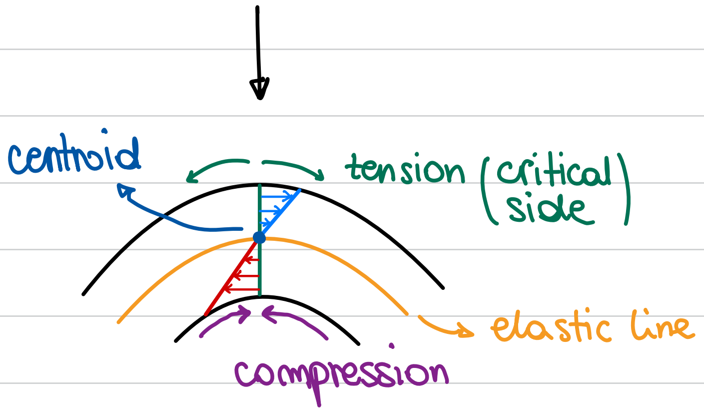
	$I_y = \frac{bh^3}{12}$ $I_z = \frac{hb^3}{12}$ $W_y = \frac{bh^2}{6}$ $W_z = \frac{hb^2}{6}$		$I_y = I_z = \frac{a^4}{12}$ $W_y = W_z = \frac{a^3}{6}$ $I_{\bar{y}} = I_{\bar{z}} = \frac{a^4}{12}$ $W_{\bar{y}} = W_{\bar{z}} = \frac{\sqrt{2}}{12}a^3 = 0{,}118\,a^3$
	$I_y = I_z = \frac{5\sqrt{3}}{16}\,R^4 = 0{,}5413\,R^4$ $W_y = \frac{5}{8}\,R^3 = 0{,}625\,R^3$ $W_z = \frac{5\sqrt{3}}{16}\,R^3 = 0{,}5413\,R^3$		$I_y = I_z = (1+2\sqrt{2})\,\frac{R^4}{6} = 0{,}638\,R^4$ $W_y = W_z = 0{,}6906\,R^3$ $I_{\bar{y}} = I_{\bar{z}} = (1+2\sqrt{2})\,\frac{R^4}{6} = 0{,}638\,R^4$ $W_{\bar{y}} = W_{\bar{z}} = 0{,}638\,R^3$
	$I_y = \frac{bh^3}{36}$ $I_z = \frac{hb^3}{48}$ $W_y = \frac{bh^2}{24}$ für $e = \frac{2}{3}h$ $W_z = \frac{hb^2}{24}$		$I_y = \frac{h^3}{36}\;\frac{b_1^2+4b_1b_2+b_2^2}{b_1+b_2}$ $W_y = \frac{h^2}{12}\;\frac{b_1^2+4b_1b_2+b_2^2}{2b_1+b_2}$ für $e = \frac{h}{3}\;\frac{2b_1+b_2}{b_1+b_2}$
	$I_y = I_z = \frac{\pi d^4}{64}$ $W_y = W_z = \frac{\pi d^3}{32}$		$I_y = I_z = \frac{\pi(D^4-d^4)}{64}$ $W_y = W_z = \frac{\pi(D^4-d^4)}{32D}$ with thin walls: $\left(\frac{s}{d_m}\right)^2 \ll 1$: $I_y = I_z = \frac{\pi d_m^3 s}{8}$, $W_y = W_z = \frac{\pi d_m^2 s}{4}$
	$I_y = \frac{\pi a^3 b}{4}$ $I_z = \frac{\pi b^3 a}{4}$ $W_y = \frac{\pi a^2 b}{4}$ $W_z = \frac{\pi b^2 a}{4}$		$I_y = \frac{\pi}{4}\,(a_1^3b_1 - a_2^3b_2)$ $W_y = \frac{\pi\left(a_1^3b_1 - a_2^3b_2\right)}{4a_1}$ With thin walls: $I_y = \frac{\pi a^2(a+3b)s}{4}$, $W_y = \frac{\pi a(a+3b)s}{4}$
	$I_y = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right)r^4 = 0{,}1098\,r^4$ $W_y = I_y/e = 0{,}1908\,r^2$ für $e = \left(1 - \frac{4}{3\pi}\right)r = 0{,}5756\,r$		$I_y = 0{,}1098(R^4-r^4) - 0{,}283R^2r^2\,\frac{R-r}{R+r}$ $W_{y1,2} = I_y/e_{1,2}$ für $e_1 = \frac{4}{3\pi}\;\frac{R^2+Rr+r^2}{R+r}$ bzw. $e_2 = R - e_1$
			$I_y = \frac{B(H^3-h^3)+b(h^3-h_1^3)}{12}$ $W_y = \frac{B(H^3-h^3)+b(h^3-h_1^3)}{6H}$
			$I_y = \frac{BH^3+bh^3}{12}$ $W_y = \frac{BH^3+bh^3}{6H}$ mit $B = B_1 + B_2$ $b = b_1 + b_2$
			$I_y = \frac{BH^3-bh^3}{12}$ $W_y = \frac{BH^3-bh^3}{6H}$ mit $b = b_1 + b_2$
			$I_y = \frac{BH^3+bh^3}{3} - (BH+bh)\,e_1^2$ mit $B = B_1 + B_2$, $b = b_1 + b_2$ $W_{y1,2} = I_y/e_{1,2}$ für $e_1 = \frac{1}{2}\;\frac{BH^2+bh^2}{BH+bh}$ bzw. $e_2 = H - e_1$

Bending

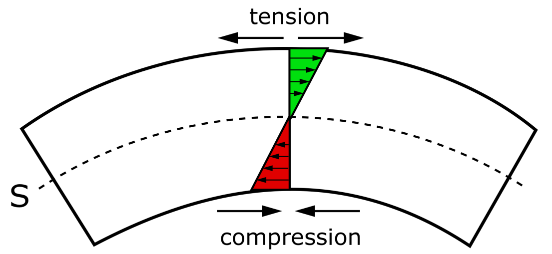
f_m = deflection

pure bending causes a deflection (f_m), a deformed elastic curve, and a bending angle





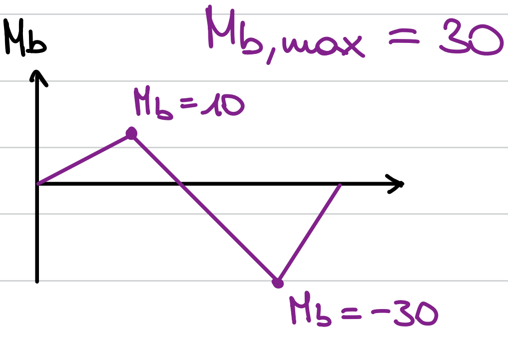
centroid: $\sigma_b = 0$!!



$$\sigma_b(y) = \frac{M_b \cdot y}{I_y} \quad \left[MPa = \frac{Nmm \cdot mm}{mm^4} \right]$$

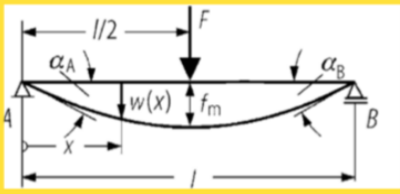
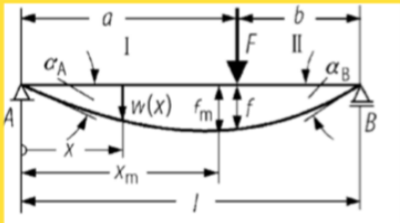
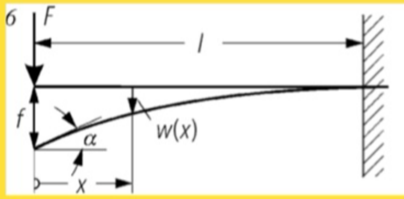
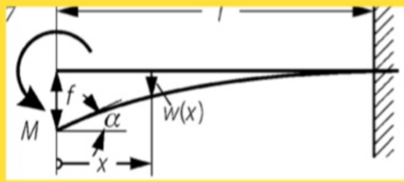
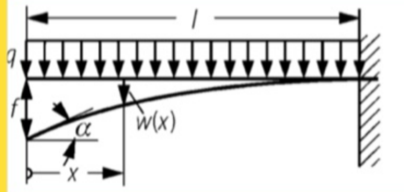
$$\sigma_{b,max} = \frac{M_{b,max} \cdot y_{max}}{I_y}$$

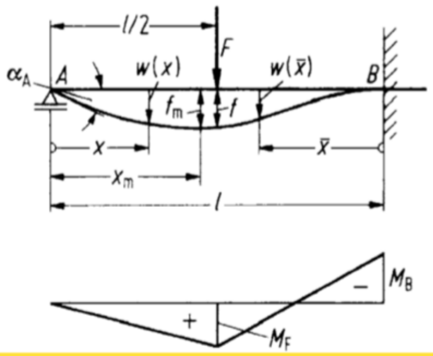
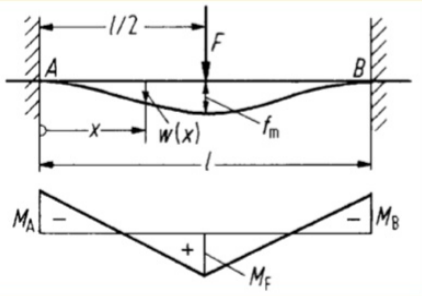
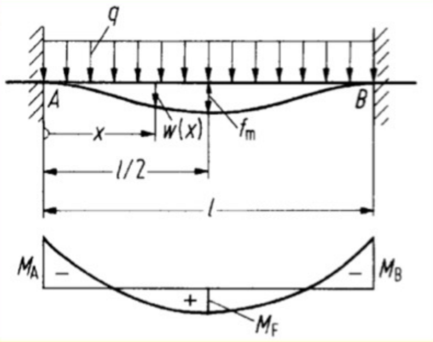
$M_{b,max}$ = abs max amplitude ⇒



Formula sheet bending

!! $\boxed{\alpha_{A,B}^{\circ} = \frac{\alpha_{A,B}\cdot 180}{\pi}}$

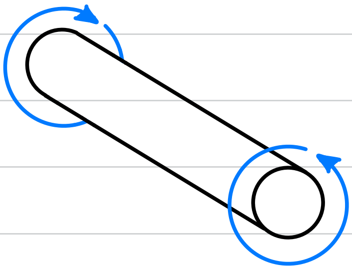
Bending case	Equation b. line	Max. deflection	↑ Curvature
	$0 \leqq x \leqq l/2;$ $w(x) = \frac{Fl^3}{48EI_y}\left[3\frac{x}{l} - 4\left(\frac{x}{l}\right)^3\right]$	$f_m = \frac{Fl^3}{48EI_y}$	$\alpha_A = \alpha_B = \frac{F\cdot l^2}{16EI_y}$
	$0 \leqq x \leqq a:$ $w_I(x) = \frac{Fab^2}{6EI_y}\left[\left(1+\frac{l}{b}\right)\frac{x}{l} - \frac{x^3}{abl}\right]$ $a \leqq x \leqq l:$ $w_{II}(x) = \frac{Fa^2b}{6EI_y}$ $\left[\left(1+\frac{l}{a}\right)\frac{l-x}{l} - \frac{(l-x)^3}{abl}\right]$	$f = \frac{Fa^2b^2}{3EI_yl}$ $a > b:\ f_m = \frac{Fb\sqrt{(l^2-b^2)^3}}{9\sqrt{3}EI_yl}$ in $x_m = \sqrt{(l^2-b^2)/3}$ $a < b:\ f_m = \frac{Fa\sqrt{(l^2-a^2)^3}}{9\sqrt{3}EI_yl}$ in $x_m = l - \sqrt{(l^2-a^2)/3}$	$\alpha_A = \frac{Fab(l+b)}{6EI_yl}$ $\alpha_B = \frac{Fab(l+a)}{6EI_yl}$
	$w(x) = \frac{Fl^3}{6EI_y}\left[2 - 3\frac{x}{l} + \left(\frac{x}{l}\right)^3\right]$	$f = \frac{Fl^3}{3EI_y}$	$\alpha = \frac{Fl^2}{2EI_y}$
	$w(x) = \frac{Ml^2}{2EI_y}\left[1 - 2\frac{x}{l} + \left(\frac{x}{l}\right)^2\right]$	$f = \frac{Ml^2}{2EI_y}$	$\alpha = \frac{Ml}{EI_y}$
	$w(x) = \frac{ql^4}{24EI_y}\left[3 - 4\frac{x}{l} + \left(\frac{x}{l}\right)^4\right]$	$f = \frac{ql^4}{8EI_y}$	$\alpha = \frac{ql^3}{6EI_y}$

Bending case	B. momentum, reaction forces	Equation b. line	Max deflection	Curvature
	$F_A = \frac{5}{16}F,\ F_B = \frac{11}{16}F$ $M_B = -\frac{3}{16}Fl$ $M_F = \frac{5}{32}Fl$	$0 \leqq x \leqq l/2:$ $w(x) = \frac{F\cdot l^3}{96EI_y}\left[3\frac{x}{l} - 5\left(\frac{x}{l}\right)^3\right]$ $0 \leqq \bar{x} \leqq l/2:$ $w(\bar{x})\ =$ $\frac{F\cdot l^3}{96EI_y}\left[9\left(\frac{\bar{x}}{l}\right)^2 - 11\left(\frac{\bar{x}}{l}\right)^3\right]$	$f = \frac{7}{768}\frac{F\cdot l^3}{EI_y}$ $f_m = \frac{F\cdot l^3}{48\sqrt{5}EI_y}$ in $x_m = \frac{l}{\sqrt{5}}$	$\alpha_A =$ $\frac{F\cdot l^2}{32EI_y}$
	$F_A = F_B = \frac{1}{2}F$ $M_A = M_B = -\frac{1}{8}Fl$ $M_F = \frac{1}{8}Fl$	$0 \leqq x \leqq l/2$ $w(x) = \frac{F\cdot l^3}{48EI_y}\left[3\left(\frac{x}{l}\right)^2 - 4\left(\frac{x}{l}\right)^3\right]$	$f_m = \frac{F\cdot l^3}{192EI_y}$	–
	$F_A = F_B = \frac{1}{2}ql$ $M_A = M_B = -\frac{1}{12}ql^2$ $M_F = \frac{1}{24}ql^2$	$w(x)\ =\ \frac{ql^4}{24EI_y}\ \cdot$ $\left[\left(\frac{x}{l}\right)^2 - 2\left(\frac{x}{l}\right)^3 + \left(\frac{x}{l}\right)^4\right]$	$f = \frac{ql^4}{384EI_y}$	–

Torsion

Pure torsion will lead to NO changes in:
- cross-section
- elastic line (neutral fiber)

Deformation in torsion is called <u>distortion</u>

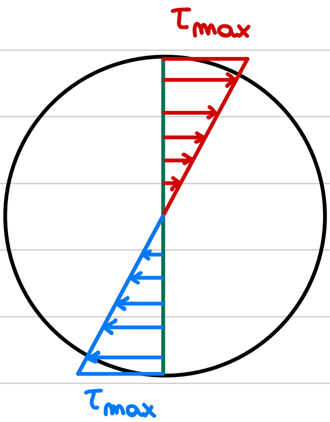


$$M_t = F \cdot d = \frac{P}{\omega} = \frac{P}{2\pi n}$$

$$\left[Nm = N \cdot m = \frac{W}{\frac{rad}{s}} = \frac{W}{\frac{1}{s}} \right]$$

$$\boxed{! \text{ if } n = rpm \rightarrow \frac{n}{60} = rps \left[{}^{1}/_{s} \right]}$$

Torsional stress

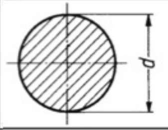
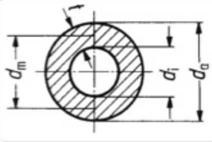
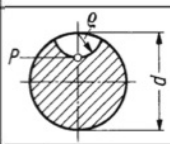
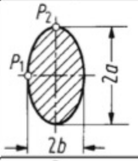
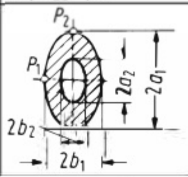
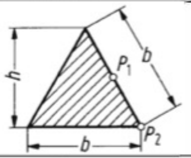
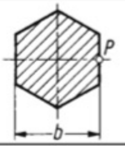
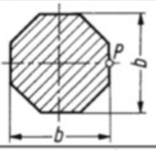
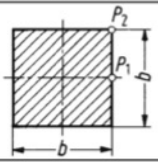
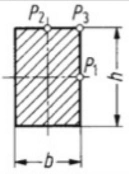


$$\tau_t = \frac{M_t}{W_t} = \frac{M_t \cdot y}{I_p}$$

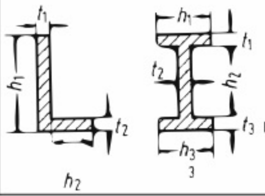
$$\left[MPa = \frac{Nmm}{mm^3} = \frac{Nmm \cdot mm}{mm^4} \right]$$

FORMULA SHEET – POLAR MOMENT OF INERTIA & Section Modulus Wt

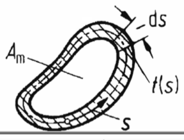
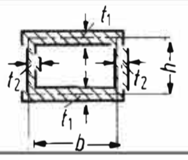
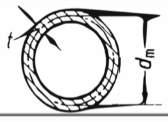
Tabelle 6. Torsionsflächenmomente I_t und -widerstandsmomente W_t

	Cross-section	I_t	W_t	Notes
1		$\frac{\pi d^4}{32} = I_p$	$\frac{\pi d^3}{16} = W_p$	τ_{max} on the circumference
2		$\frac{\pi\left(d_a^4-d_i^4\right)}{32} = I_p$ Für geringe Wanddicken, d. h. $\left(\frac{t}{d_m}\right)^2 \ll 1$: $\pi d_m^3 t/4$	$\frac{\pi\left(d_a^4-d_i^4\right)}{16 d_a} = W_p$ Für geringe Wanddicken, d. h. $\left(\frac{t}{d_m}\right)^2 \ll 1$: $\pi d_m^2 t/2$	τ_{max} on the circumference
3		$\frac{\pi d^4}{32} = I_p$	$\frac{W_p}{\lambda} = \frac{\pi d^3}{16\lambda}$ $\lambda = \frac{2-\xi}{1-2\xi^2+(16/3\pi)\xi^3}$ Für kleine ξ : $\lambda \approx 2$	τ_{max} at the base of the notch (at P) $\xi = \frac{\varrho}{d/2}$
4		$\frac{\pi a^3 b^3}{a^2+b^2} = \frac{\pi n^3 b^4}{n^2+1}$	$\frac{\pi a b^2}{2} = \frac{\pi n b^3}{2}$	Prerequisite: $a/b = n \geq 1$ τ_{max} at P_1 in $P_2 : \tau_2 = \tau_{max}/n$
5		$\frac{\pi n^3\left(b_1^4-b_2^4\right)}{n^2+1}$	$\frac{\pi n\left(b_1^4-b_2^4\right)}{2b_1}$	Prerequisite: $a_1/b_1 = a_2/b_2 = n \geq$ 1 τ_{max} at P_1 in $P_2 : \tau_2 = \tau_{max}/n$
6		$\frac{b^4}{46{,}19} \approx \frac{h^4}{26}$	$\frac{b^3}{20} \approx \frac{h^3}{13}$	τ_{max} iat the mid point of the side (P_1) at the corners (P_2): $\tau_2 = 0$
7		$0{,}133 b^2 A = 0{,}115 b^4$	$0{,}217 b A = 0{,}188 b^3$	τ_{max} at the mid point of the side (P)
8		$0{,}130 b^2 A = 0{,}108 b^4$	$0{,}223 b A = 0{,}185 b^3$	τ_{max} at the mid point of the side (P)
9		$0{,}141 b^4$	$0{,}208 b^3$	τ_{max} at the mid point of the side (P_1) at the corners (P_2): $\tau_2 = 0$
10		$c_1 h b^3 = c_1 n b^4$	$c_2 h b^2 = c_2 n b^3$	Prerequisite: $h/b = n \geq 1$ τ_{max} in P_1 In $P_2 : \tau_2 = c_3 \tau_{max}$ In $P_3 : \tau_3 = 0$

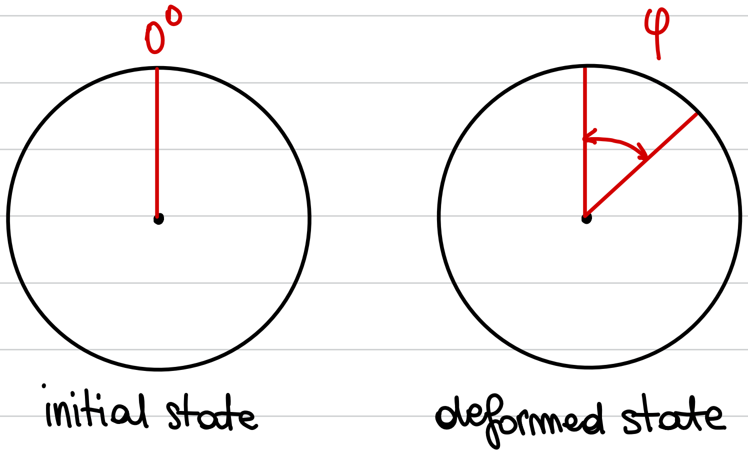
$n = h/b$	1	1,5	2	3	4	6	8	10	∞
c_1	0,141	0,196	0,229	0,263	0,281	0,298	0,307	0,312	0,333
c_2	0,208	0,231	0,246	0,267	0,282	0,299	0,307	0,312	0,333
c_3	1,000	0,858	0,796	0,753	0,745	0,743	0,743	0,743	0,743

	Cross-section	I_t	W_t	Notes
Thin walled profiles 11		$\frac{\eta}{3} \sum h_i t_i^3$	I_t/t_{max}	Prerequisite: $h_i/t_i \gg 1$ τ_{max} in the centre of the long side of the rectangle with t_{max}

Profil	L	⊏	⊥	I	I PB	+
η	0,99	1,12	1,12	1,31	1,29	1,17

	Cross-section	I_t	W_t	Notes
12	Thin-walled hollow cross-sections 	$\frac{4 A_m^2}{\oint ds/t(s)}$ For constant wall-thickness t: $4 A_m^2 t/U$	$2 A_m t_{min}$ For constant wall-thickness t: $2 A_m t$	A_m: Area of hollow part of the cross-section. U: Circumference of interiour line, τ_{max} **at the point where,** wo $t = t_{min}$. Es gilt: $\tau(s) \cdot t(s) = M_t / 2 A_m = \text{const}$
12a		$\frac{4(bh)^2}{2(b/t_1+h/t_2)}$	$2 b h t_{min}$	τ_{max} **where** $t = t_{min}$
12b		$\pi d_m^3 t/4$	$\pi d_m^2 t/2$	

Torsional deformation (φ)



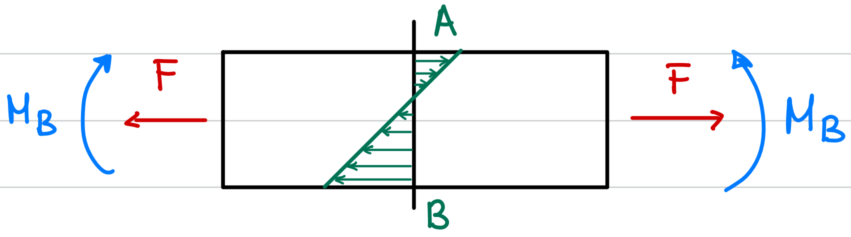
$$\varphi(x) = \frac{M_t(x) \cdot l}{I_p \cdot G} \cdot \frac{180^\circ}{\pi} \quad [^\circ]$$

where :

G = shear modulus of a material [MPa]

$$\varphi_{TOT} = \frac{1}{G} \sum_i \frac{M_{t,i} \cdot l_i}{I_{p,i}} \xrightarrow{\cdot \frac{180}{\pi}} [^\circ]$$

Combining stress



Combining tensile with bending stress

at A: $\sigma_b - \sigma_t$

at B: $-\sigma_b - \sigma_t$