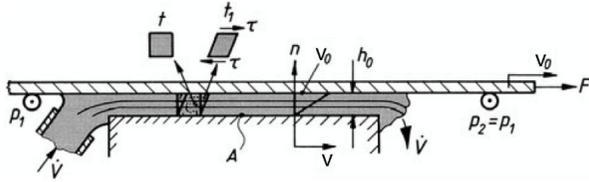


I. Viscous flow

Symbology

Symb	Name	Units	Symb	Name	Units
ν	Kinematic viscosity	m^2/s	η	Dynamic viscosity	$Pa \cdot s$
τ	Shear stress	Pa	dn	Normal distance	m
du/dy	Rate of strain	$[-]$	e_{Diss}	Spec. diss. power	W/m^3

1. Flows with friction



$$F = \eta \frac{v_0 \cdot A}{h_0} \implies \tau = \frac{F}{A} = \eta \frac{v_0}{h_0} = \eta \frac{dv}{dn}$$

For pipes:

$$F_\tau = \tau A \implies M_\tau = F_\tau y = \tau \cdot l \cdot 2\pi r_0^2$$

1.1 Friction law

$$\tau = -\eta \frac{dv}{dn} ; \quad \dot{e}_{Diss} = \eta \left(\frac{dv}{dn} \right)^2$$

2. Viscosity

If $\eta = 0$, the fluid is called inviscid.

No slip condition: velocity of a particle closest to the wall has velocity equal to the wall velocity.

2.1 Reynolds Number

$$Re = \frac{\rho v L}{\eta} = \frac{v L}{\nu} ; \quad \nu = \frac{\eta}{\rho}$$

2.2 Newtonian fluids

- Rate of deformation du/dy (velocity gradient) is linearly proportional to the shear stress τ
- The constant of proportionality is the viscosity (slope)
- $\eta = \frac{\tau}{du/dy} = \text{constant}$

2.3 Non-Newtonian fluids

Non-linear relation between shear stress and deformation rate due to non-constant viscosity.

$$\tau = \eta \left(\frac{\partial u}{\partial y} \right) \cdot \frac{\partial u}{\partial y}$$

2.3.1 Dilatant (shear thickening) fluids

Viscosity increases with increasing strain rate.

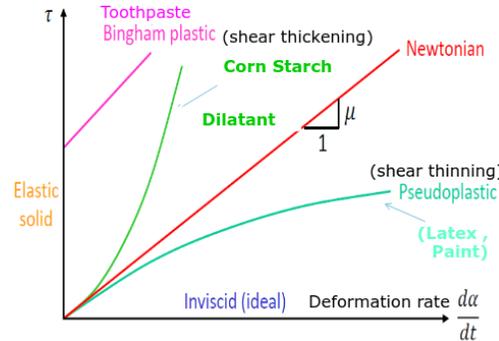
2.3.2 Plastic or pseudo-plastic (shear thinning) fluids

Viscosity decreases with increasing strain rate.

2.3.3 Bingham medium

Flow occurs only after a yield stress τ_0 is reached.

2.4 Graphical representation



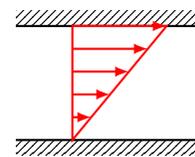
3. Flow profiles

3.1 Couette flow profiles (constant velocity gradient)

$$\frac{\partial u(y)}{\partial y} = \frac{v_p}{s}$$

$$e_{Diss} = \eta \left(\frac{\partial u(y)}{\partial y} \right)^2 = \eta \left(\frac{v_p}{s} \right)^2$$

$$P_{Diss} = \int_V e_{Diss}(y) dV$$



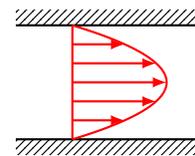
3.2 Poiseuille flow profiles (parabolic velocity gradient)

$$\frac{\partial u(y)}{\partial y} = \frac{4v_{max}}{s} \left(1 - \frac{2y}{s} \right)$$

$$v(r) = ar^2 + br + c$$

$$e_{Diss} = \eta \frac{16v_{max}^2}{s^2} \left(1 - \frac{2y}{s} \right)^2$$

$$P_{Diss} = \frac{16\eta A v_{max}^2}{3s}$$



4. Laminar pipe flow

4.1 $p_1 > p_2$

$$\frac{\partial p}{\partial z} = \frac{p_2 - p_1}{L}$$

4.2 Constant linear flow profiles

$$\dot{V} = v_m A = v_m R^2 \pi$$

4.3 Parabolic flow profiles

$$\dot{V} = \int v(r) dA = \frac{v_{max} \cdot R^2 \pi}{2}$$

$$v(r) = v_{max} \left(1 - \frac{r^2}{R^2} \right)$$

$$\frac{dv}{dr} = -2v_{max} \frac{r}{R^2}$$

$$P_{Diss} = \int_0^L \int_0^{2\pi} \int_0^R \eta \left(-2v_{max} \frac{r^2}{R^2} \right) \cdot r dr d\varphi dx$$

$$P_{Diss} = 2\pi L \int_0^R \eta \left(-2v_{max} \frac{r^2}{R^2} \right) \cdot r dr = 2\pi L \eta v_{max}^2 = 8\pi L \eta v_m^2$$

5. Flows in gaps and bearings

5.1 Assumptions in gaps and bearings

- Newtonian fluid with constant viscosity η
- Incompressible fluid
- Very low Reynolds number $Re < 2000$
- Laminar flow
- Balance of pressure and viscous forces
- No slip condition at the walls
- No acceleration (negligible)
- Pressure is uniform across the thickness $\partial p / \partial y \approx 0$

5.2 Equilibrium of forces

$$\sum F_x = 0 \implies pA_p - (p + dp)A_p + \tau A_\tau + (\tau + d\tau)A_\tau = 0$$

$$A_p = dy \cdot b ; \quad A_\tau = dx \cdot b ; \quad \tau = \eta \frac{dv}{dy} \rightarrow \frac{d\tau}{dy}$$

$$\boxed{\eta \frac{d^2 v(x, y)}{dy^2} = \frac{dp(x)}{dx} = p'}$$

5.3 Basic velocity equation

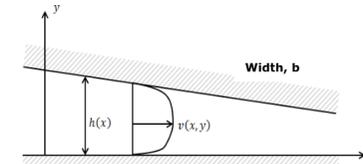
$$v(x, y) = \frac{1}{\eta} \cdot p' \cdot \frac{y^2}{2} + c(x)y + d(x)$$

5.3.1 Gap flow (walls not moving)

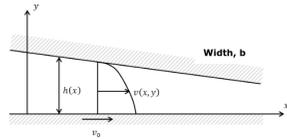
$$v(x, y) = \frac{p'(x)}{2\eta} [y^2 - h(x)y]$$

$$c(x) = -\frac{h(x)}{2\eta} \cdot \frac{dp(x)}{dx}$$

$$d(x) = 0$$



5.3.2 Plain bearing / Creeping gap flow (one wall moving)



Assumption for $Re \ll 1$: $\eta \frac{d^2v}{dy^2} = \frac{dp}{dx} = p'$

$$v(x, y) = \frac{p'(x)}{2\eta} [y^2 - h(x)y] + \frac{v_0}{h(x)} [h(x) - y]$$

$$c(x) = -\frac{h(x)}{2\eta} \cdot \frac{dp(x)}{dx} - \frac{v_0}{h(x)}$$

$$d(x) = v_0$$

For the simplest, parabolic velocity profile case:

$$h(x) = h_0 \quad ; \quad v_0 = 0 \quad ; \quad p' = \frac{dp(x)}{dx} = \frac{\Delta p}{\Delta L} = -\frac{12\eta v_m}{h_0^2}$$

$$v(y) = \frac{p'}{2\eta} (y^2 - h_0 y) \implies \frac{\partial v}{\partial y} = 0 = \frac{p'}{2\eta} (2y - h_0)$$

$$y = \frac{h_0}{2} \implies v_{max} = v \left(y = \frac{h_0}{2} \right) = -\frac{h_0^2 p'}{8\eta} = \frac{3}{2} v_m$$

$$v_m = -\frac{\Delta p \cdot h_0^2}{12\eta L}$$

$$\dot{V} = b \int_0^{h_0} v(y) dy = -b \frac{p' h_0^3}{12\eta} = \frac{h_0^2 b \Delta p}{6\eta}$$

General sliding bearings:

$$\dot{V} = b \int_0^{h(x)} \left[\frac{p'}{2} \eta (y^2 - h(x)y) + \frac{v_0}{h(x)} (h(x) - y) \right] dy$$

$$\dot{V} = -b \frac{p' h(x)^3}{12\eta} + b \frac{v_0 h(x)}{2} = \phi$$

$$p' = 6\eta v_0 \left(\frac{1}{h(x)^2} - \frac{\dot{V}}{b h(x)^3} \right)$$

5.3.3 General form of the volumetric flow rate

$$v(x, y) = \frac{1}{\eta} p' \frac{y^2}{2} + c(x) + d(x)$$

$$\dot{V} = b \int_0^{h(x)} v(x, y) dy = b \left[-\frac{h^3(x)}{6\eta} p' + c(x) \frac{h^2(x)}{2} + d(x) h(x) \right] = \phi$$

$$p' = \frac{6\eta}{h(x)^3} \left(\frac{\dot{V}}{b} - c(x) \frac{h(x)^2}{2} - d(x) h(x) \right)$$

5.3.4 Possible simplifications

If $h \ll D$:

$$A = \int_A r dr d\varphi \approx \pi D h$$

5.4 Specific dissipation power

$$P = \tau A v = \eta \frac{\partial v}{\partial n} A v \quad ; \quad dP = dF dv$$

$$dF_\tau = d\tau dA = \eta \frac{dv}{dy} dx dz$$

$$e_{Diss} = \frac{dP_\tau}{dV} = \frac{\eta \frac{dv}{dy} dx dz dv}{dx dy dz} = \eta \left(\frac{dv}{dy} \right)^2 = \eta \left(\frac{\partial v}{\partial y} \right)^2$$

$$P_{Diss} = \int_V e_{Diss} dV = \iiint_V e_{Diss} dx dy dz = M \cdot \omega$$

5.4.1 Pressure loss according to Bernoulli

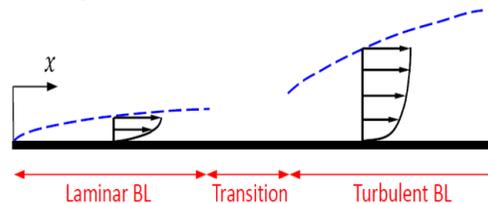
$$p_1 - p_2 = \rho \frac{P_{Diss}}{\dot{m}} = \frac{P_{Diss}}{\dot{V}} = \frac{8\pi L \eta v_m}{R^2}$$

II. Boundary layers

Symb	Name	Units	Symb	Name	Units
v_∞	free-stream velocity	m/s	$v(y)$	local velocity in boundary layer	m/s
δ	boundary layer thickness	m	δ^*	displacement thickness	m
θ	momentum thickness	m	τ_w	wall shear stress	Pa
ν	kinematic viscosity	m ² /s	Re_x	Reynolds number at position x	-
x	distance from leading edge	m	\dot{V}_{BL}	boundary-layer volume flow rate	m ² /s

6. Laminar vs. Turbulent boundary layers

6.1 Critical Reynolds number

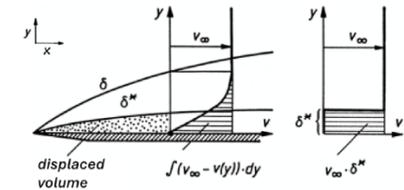


Reynolds number for the BL on a flat plate:

$$Re_x = \frac{v_\infty x}{\nu} = \frac{v_\infty \cdot l_{char}}{\nu}$$

Re **increases** with the distance from the plate entry.
Transition to turbulent: $Re_x \geq 5e5$ (usually $3.2e5$ to $3e6$)

6.2 Flat plate boundary layer



where:

- $\delta(x) \approx \sqrt{x}$
- $\tau(x) \approx 1/\delta(x) = 1/\sqrt{x}$
- $\dot{V}_{BL} = v_\infty \cdot \delta^*(x)$