Linear Algebra
HSLU, Semester 4

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Contents

1 Vectors

1.1 Linear combination

A sum of scalings of vectors is called a linear combination of the vectors.

Let $\vec{u}$, $\vec{v}$ be vectors, and a, b be scalars, $a, b \in \mathbb{R}$, then:

$$\boxed{a \cdot \vec{u} + b \cdot \vec{v}}$$

Generalizing this to a set of vectors $\vec{u}_1, \ldots, \vec{u}_n$, and scalars $a_1, \ldots, a_n$, we have:

$$\boxed{\sum_{i=1}^{n} a_i \cdot \vec{u}_i}$$

1.2 Cross product (vector product)

The cross product of two vectors $\vec{u}$ and $\vec{v}$ is a vector $\vec{w}$ that is perpendicular to both $\vec{u}$ and $\vec{v}$, and has a magnitude equal to the area of the parallelogram formed by $\vec{u}$ and $\vec{v}$.

Let $\vec{u} = (x, y, z)$ and $\vec{v} = (t, s, q)$

$$\boxed{\vec{u} \times \vec{v} = (yq - sz,\ -(xq - tz),\ xs - yt)}$$

1.3 Unit vectors

Components of the unit vectors of the vector basis are as follow:

$$\boxed{\vec{i} = \vec{e}_x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{j} = \vec{e}_y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{k} = \vec{e}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

with their norms being $\|\vec{i}\| = \|\vec{j}\| = \|\vec{k}\| = 1$

2 Gaussian elimination

We want to archieve a stair of zeros on the bottom left of the matrix:

$$\left[\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right]$$

where $*$ is a non-zero number.

This also applies to higher dimensions, for example:

$$\left[\begin{array}{ccccccc|c} * & * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \end{array}\right]$$

2.1 Solution cases

2.1.1 1 possible solution

$$\left[\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right] \qquad \begin{cases} x = * \\ y = * \\ z = * \end{cases}$$

2.1.2 Infinitely many solutions

$$\left[\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{cases} x = * \\ y = * \\ z = t \in \mathbb{R} \end{cases}$$

2.1.3 No solution

$$\left[\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{array}\right] \qquad \begin{cases} x = * \\ y = * \\ 0 = * \to z \in \varnothing \end{cases}$$

3 dunno I forgot to take notes

4 Inverse and Determinant

Let A be a square matrix. The inverse of A, denoted by A^{-1}, satisfies the following:

$$\boxed{A \cdot A^{-1} = I_n}$$

where I_n is the identity matrix $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$

4.1 Determinant

For $M \in \mathbb{R}^{2\times 2}$, $\det M = ad - bc$

For $M \in \mathbb{R}^{3\times 3}$, use Sarrus rule:

$$\left[\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right]$$

$$\boxed{\det M = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}}$$

For $M \in \mathbb{R}^{n\times n}$, $n > 3$, apply row enchelon and multiply the diagonal pivots.

4.2 Inverse

Let M be a 2×2 matrix and $a_{\substack{1\le i\le m \\ 1\le j\le n}}$

$$\boxed{M^{-1} = \frac{1}{\det M}\,\mathrm{adj}(M), \quad \det M \neq 0}$$

where:

$$\boxed{A \in \mathbb{R}^{2\times 2},\ \mathrm{adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

For $B \in \mathbb{R}^{3\times 3}$, apply Gauss - Jordan elimination $[A|I_n] \Leftrightarrow [I_n|A^{-1}]$

4.3 Linear systems

$$\boxed{Ax = b \Longleftrightarrow x = A^{-1} \bullet b}$$

5 Add something here

6 Linear independency

6.1 with 1 vector

If we only have one vector $\vec{v_k}$ at any $\mathbb{R}^k$, then it is independent.

7 Linearità

$$\boxed{f(\alpha v + \beta w) = \alpha f(v) + \beta f(w)}$$

7.1 Rotation

7.1.1 Clockwise

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

7.1.2 Counter-clockwise

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

8 Inverse map

$L(v) = w$ is invertible if and only if v is a basis (linear independent) and w is also linear independent.

$$\boxed{w = L(v) \iff v = L^{-1}(w)}$$

If $L : \mathbb{R}^n \mapsto \mathbb{R}^n$ has a matrix A, then

$$L^{-1}(x) = A^{-1}x$$

then L is invertible iff $\det A \neq 0$

Follows that, for $L(e_1) = v$ and $L(e_2) = w$, L is invertible iff v, w are linearly independent, as $e_1,\ e_2$ are linearly independent.

To calculate $L^{-1} \in \mathbb{R}^2$:

$$\boxed{L^{-1} = \frac{1}{\det L} \operatorname{adj} L = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

8.1 Kernel and image

Let A be a matrix, then:

$$\ker A = \{x \in \mathbb{R}^n \mid Ax = 0\}$$
$$\operatorname{im} A = \{Ax \mid x \in \mathbb{R}^n\} \Longrightarrow \operatorname{im} A = \operatorname{span}\{a_1, \dots, a_n\}$$