1 Preambule

Theory box

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Formula box

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Lab/examples box

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2 Fluids as energy carriers

2.1 Fluid state variables and properties

Formulas

2.1.1 State variables

Density

$$\rho \triangleq \frac{m}{V} \left[\frac{kg}{m^3}\right] \tag{1}$$

Specific volume

$$v \triangleq \frac{V}{m} = \frac{1}{\rho} \left[\frac{m^3}{kg}\right] \tag{2}$$

2.1.2 Viscosity

Kinematic viscosity

$$\nu \triangleq \frac{\eta}{\rho} \left[\frac{m^2}{s}\right] \tag{3}$$

Dynamic viscosity

$$\eta \triangleq \nu \cdot \rho \left[Pa \cdot s = \frac{Ns}{m^2} = \frac{kg}{m \cdot s}\right] \tag{4}$$

2.1.3 Real and ideal fluid

Real fluid	**Ideal fluid**
variable density ($\Delta\rho \neq 0$)	incompressible ($\Delta\rho = 0$)
friction ($\eta > 0, \nu > 0$)	frictionless ($\eta = 0, \nu = 0$)

2.1.4 Compressibility

Mach number

$$M \triangleq \frac{u}{c} \tag{5}$$

where:

- M is the Mach number [-]
  $M \lesssim 0.3$: incompressible flow
- u is the flow velocity [m/s]
- c is the speed of sound in the fluid [m/s]

and:

- $c_{\text{w}}^{20°} = 1484$ m/s
- $c_{\text{a}}^{20°} = 343$ m/s

2.2 Laminar and turbulent flow

Reynolds number

$$Re = \frac{v \cdot L}{\nu} = \frac{\rho \cdot v \cdot L}{\eta} \, [-] \tag{6}$$

where:

- v is the mean flow velocity [m/s]
- L is the characteristic length [m]

Re values

- $Re < 2000$: laminar flow
- $Re \simeq 2300$: critical point
- $2000 < Re < 4000$: transitional regime
- $Re \geq 4000$: turbulent flow

2.3 Pressure and velocity

Pressure

2.3.1 Total pressure

Added to the static pressure p_{stat}, there is also the dynamic pressure p_{dyn} and the total pressure p_{tot}:

$$p_{\text{tot}} = p_{\text{stat}} + p_{\text{dyn}} = \rho \left(gh + \frac{v^2}{2}\right) \tag{7}$$

2.3.2 Absolute pressure

Absolute pressure p_{abs} refers to the pressure in a vacuum $p_{\text{vaacum}} = 0$ Pa, while relative pressure p_{rel} can refer to any chosen reference pressure p_{ref}.

$$p_{\text{rel}} = p_{\text{abs}} - p_{\text{ref}} \iff p_{\text{abs}} = p_{\text{rel}} + p_{\text{ref}} \geq 0 \tag{8}$$

2.3.3 Velocity

Velocity is a vector quantity:

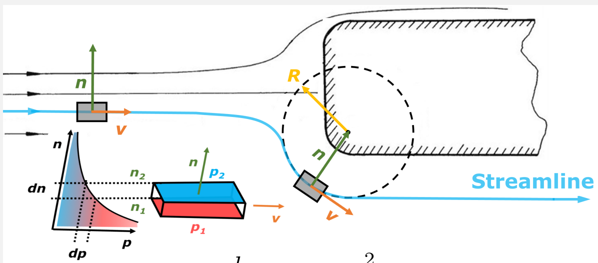
$$\vec{v} = (v_x v_y v_z) \tag{9}$$

The magnitude is given by:

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} \tag{10}$$

2.4 Curvature pressure formula

Deflection motion of a fluid element around a blunt body

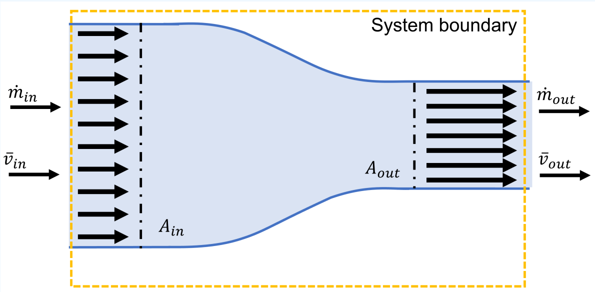


$$\frac{dp}{dn} = -\rho \cdot \frac{v^2}{R} \tag{11}$$

3 Mass conservation

3.1 Continuity equation / Mass conservation

Continuity equation



3.1.1 Steady mass-flow

$$\dot{m}_{\text{in}} = \dot{m}_{\text{out}} \tag{12}$$

3.1.2 Incompressible fluid

$$\dot{m} = \rho \dot{V} \implies \dot{V}_{\text{in}} = \dot{V}_{\text{out}} \tag{13}$$

3.1.3 Streamline theory

$$\dot{V} = \bar{v}\, A \implies \bar{v}_{\text{in}}\, A_{\text{in}} = \bar{v}_{\text{out}}\, A_{\text{out}} \tag{14}$$

4 Energy conservation

4.1 Fluid mechanical energy conservation

Derivation of the Bernoulli equation

$$\dot{m}_1 \left(\frac{p_1}{\rho} + \frac{v_1^2}{2} + gz_1 \right) = \dot{m}_2 \left(\frac{p_2}{\rho} + \frac{v_2^2}{2} + gz_2 \right) \tag{15}$$

This derivation is based on the assumption that the system has:

- steady flow
- ideal fluid
- adiabatic process
- no work in or out of the system
- 1D streamline flow

4.1.1 Energy flow

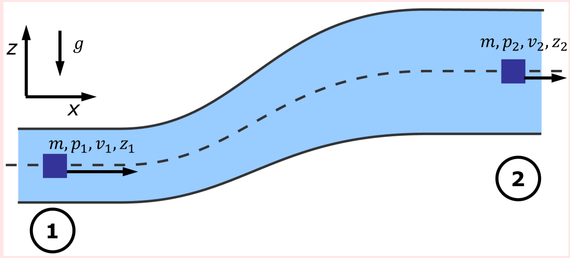
$$\frac{dE}{dt} = \underbrace{\sum P + \sum \dot{Q}}_{\substack{\text{Energy flow} \\ \text{across system boundary}}} + \underbrace{\sum_{in} \left[\dot{m}^{\swarrow} \cdot \left(h^{\swarrow} + \frac{v^{2\swarrow}}{2} + gz^{\swarrow} \right) \right]}_{\substack{\text{Energy transfer} \\ \text{mass in}}} - \underbrace{\sum_{out} \left[\dot{m}^{\nearrow} \cdot \left(h^{\nearrow} + \frac{v^{2\nearrow}}{2} + gz^{\nearrow} \right) \right]}_{\substack{\text{Energy transfer} \\ \text{mass out}}} \tag{16}$$

4.1.2 Outflow formula according to Torricelli

$$gz_1 = \frac{v_2^2}{2} \implies v_2 = \sqrt{2g\Delta z} \tag{17}$$

4.2 Bernoulli equation

Specific energy equation



$$\frac{p_1}{\rho} + \frac{v_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{v_2^2}{2} + gz_2 = \text{const.} \left[\frac{J}{kg} \right] \tag{18}$$

4.2.1 Alternative forms

Pressure equation

$$p_1 + \frac{\rho v_1^2}{2} + \rho g z_1 = p_2 + \frac{\rho v_2^2}{2} + \rho g z_2 = \text{const.}\, [Pa] \tag{19}$$

Height equation

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 = \text{const.}\, [m] \tag{20}$$

True energy equation

The Bernoulli equation states that the sum of these energies is constant along a streamline.

4.2.2 Pressure energy

$$E_p = m \cdot \frac{p}{\rho}\, [J] \tag{21}$$

4.2.3 Kinetic energy

$$E_{\text{kin}} = m \cdot \frac{v^2}{2}\, [J] \tag{22}$$

4.2.4 Potential energy

$$E_{\text{pot}} = m \cdot g \cdot z\, [J] \tag{23}$$

4.2.5 Energy conservation

$$E_{p,1} + E_{\text{kin},1} + E_{\text{pot},1} = E_{p,2} + E_{\text{kin},2} + E_{\text{pot},2}$$

$$m \left(\frac{p_1}{\rho} + \frac{v_1^2}{2} + gz_1 \right) = m \left(\frac{p_2}{\rho} + \frac{v_2^2}{2} + gz_2 \right) \tag{24}$$

4.3 Hydrostatics

Fundamental law of hydrostatics

$$p = p_0 + \rho g h = \text{const.}\, [\text{Pa}] \tag{25}$$

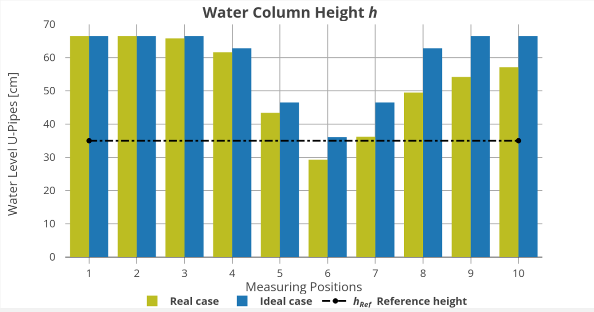
derived from:

$$p = p_0 + \frac{F_g}{A} = p_0 + \frac{mg}{A} = p_0 + \frac{\rho h A g}{A} \tag{26}$$

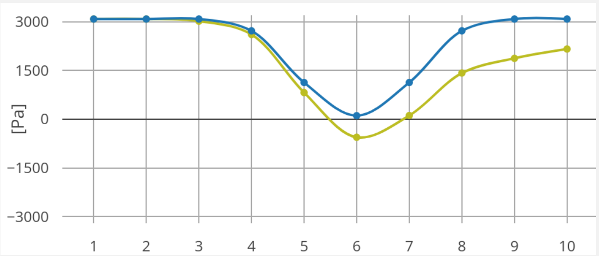
Venturi effect

4.4 Venturi effect experiment

4.4.1 Height – pressure difference at $\dot{V} = 6$ l/s

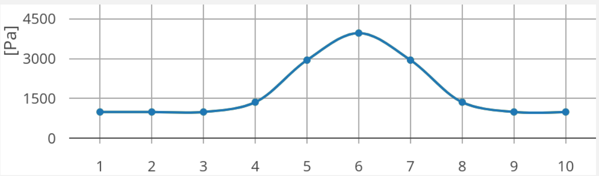


4.4.2 Relative static pressure p_{rel}



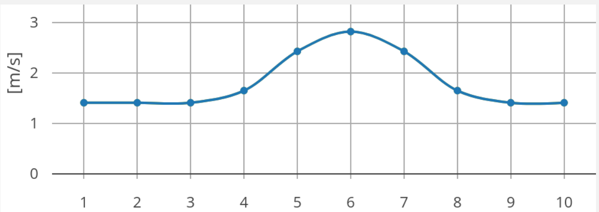
$$p_{\text{rel}} = p_{\text{hydro}} = \rho g \left(h - h_{\text{ref}}\right) \tag{27}$$

4.4.3 Dynamic pressure p_{dyn}



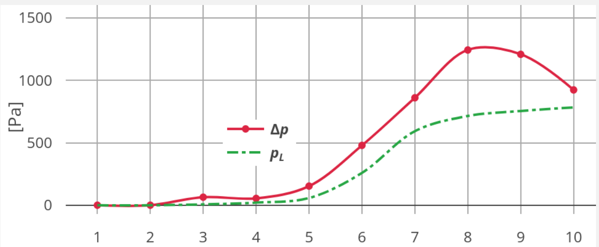
$$p_{\text{dyn}} = \rho \frac{v^2}{2} \tag{28}$$

4.4.4 Dynamic pressure v



$$v = \frac{\dot{V}}{A} \tag{29}$$

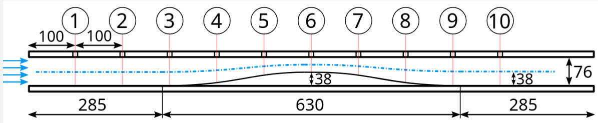
4.4.5 Pressure difference Δp



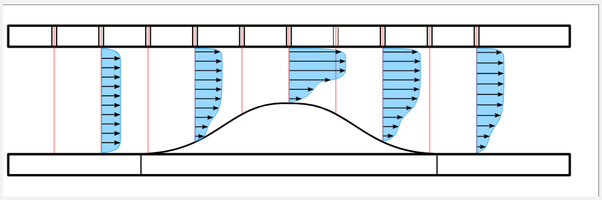
$$\Delta p = p_{\text{NoFric}} - p_{\text{real}} \Longrightarrow p_V \sim v^2 \tag{30}$$

Venturi effect

Measurement points



Measurement shear flow

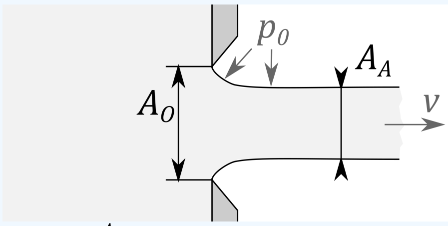


Just to not forget

$$A_{\text{pipe}} = D^2 \pi = \frac{r^2 \pi}{4} \Longleftrightarrow D = 2\sqrt{\frac{A}{\pi}} \tag{31}$$

4.5 Contraction coefficient

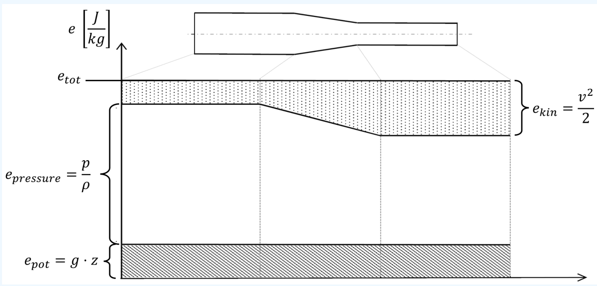
Outflow contraction coefficient α



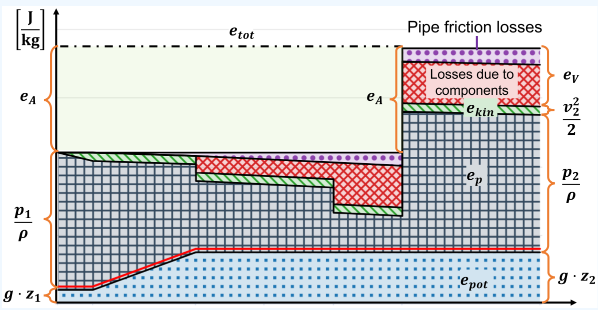
$$\alpha = \frac{A_{\text{actual}}}{A_{\text{opening}}} = \frac{\pi}{2+\pi} \approx 0.611[-] \tag{32}$$

4.6 Energy line diagram

Ideal fluid energy line diagram



Extended energy line diagram



4.7 Extended Bernoulli equation

Extension of the Bernoulli equation

$$\frac{p_1}{\rho} + \frac{v_1^2}{2} + gz_1 + e_A = \frac{p_2}{\rho} + \frac{v_2^2}{2} + gz_2 + e_V \left[\frac{J}{kg}\right]$$
$$E_{p,1} + K_1 + U_1 + E_A = E_{p,2} + K_2 + U_2 + E_V \; [J] \quad (33)$$

4.7.1 Additional terms

Work term e_A

$$e_A = \frac{p_A}{\rho} = gz_A = \frac{E_A}{m} = \frac{P_A}{\dot{m}} \left[\frac{J}{kg}\right] \quad (34)$$

where:
e_A: work term [J/kg] E_A: energy difference [J]
p_A: pressure diff [Pa] P_A: power difference [W]
z_A: height difference [m]

If energy is added to the fluid along a streamline from point 1 to point 2 (eg. a pump), the total energy at point 2 becomes higher than at point 1.

Sign convention
$\mathbf{e_A > 0}$: work is done on the fluid
→ energy is added to the fluid (eg. pump);

$\mathbf{e_A < 0}$: work is done by the fluid
→ energy is extracted from the fluid (eg. turbine).

Pump and turbine work Y

In the pressure equation, the pressure p_A increase (or decrease with a turbine) can be read directly at the working term, hence:

$$e_w = Y = \frac{W_A}{\dot{m}} = \frac{E_A}{m} = H \cdot g = \frac{p_A}{\rho} \left[\frac{J}{kg}\right] \quad (35)$$

The hydraulic power P_{hyd} is then given by:

$$P_{\text{hyd}} = \dot{m} \cdot Y = \dot{V} \cdot \rho \cdot Y = \rho \cdot \dot{V} \cdot g \cdot H \; [W] \quad (36)$$

Specific loss term e_V

$$e_V = \frac{p_V}{\rho} = gz_V = \frac{E_V}{m} = \frac{P_V}{\dot{m}} \left[\frac{J}{kg}\right] \quad (37)$$

where:
e_V: loss term [J/kg] E_V: energy loss [J]
p_V: pressure diff [Pa] P_V: power loss [W]
z_V: height loss [m]

The effects of a viscous fluid along a stramline from point 1 to point 2 are taken into account by e_V.

Pressure loss Δp_V

$$\Delta p_V = e_V \cdot \rho = \frac{E_V \cdot \rho}{m} = g \cdot z_V \cdot \rho = \zeta \cdot \rho \cdot \frac{v^2}{2} \; [Pa] \quad (38)$$

4.8 Loss behavior in turbolent flows

Zeta value

$$\zeta = \frac{2 \cdot \Delta p_V}{\rho \cdot v^2} \quad (39)$$

Total pressure loss

If multiple losses occur in a system due to sequentially connected hydraulic components, the ttal loss $\Delta p_{V,\text{tot}}$ is given by the sum of the individual losses:

$$\Delta p_{V,\text{tot}} = \sum_i \Delta p_{V,i} = \sum_i \zeta_i \cdot \rho \cdot \frac{v_i^2}{2} \; [Pa] \quad (40)$$

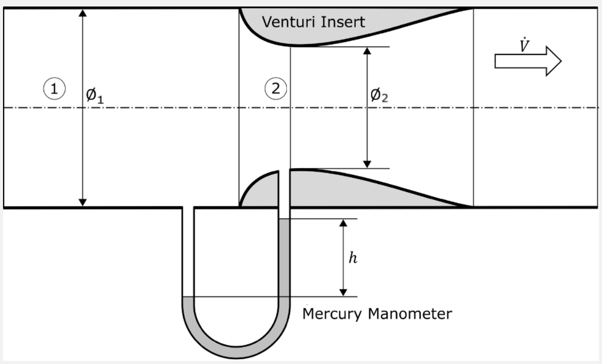
$$\Delta p_{V,\text{tot}} = \rho \cdot \frac{v^2}{2} \cdot \sum_i \zeta_i = \rho \cdot \frac{v^2}{2} \cdot \zeta_{\text{tot}} \; [Pa] \quad (41)$$

Pressure head (prevalenza)

The pressure head H is the (energy) height corresponding to its specific potential energy e_A:

$$H = \frac{e_A}{g} = \frac{\Delta p_A}{\rho \cdot g} \; [m] \quad (42)$$

U-Tube manometer



$$h = \frac{\rho \left(v_2^2 - v_1^2\right)}{2g \left(\rho_{\text{Hg}} - \rho_w\right)} \quad (43)$$

4.9 Efficiency

Efficiency factor η

$$\eta = \frac{P_{\text{out}}}{P_{\text{in}}} = \frac{\text{Benefit}}{\text{Effort}} \quad (44)$$

$$\eta_{\text{hyd}} = \frac{P_{\text{real}}}{P_{\text{ideal}}} = \frac{\dot{m} \cdot e_{\text{real}}}{\dot{m} \cdot e_{\text{ideal}}} = \frac{e_A - e_V}{e_A}$$
$$\eta_{\text{hyd}} = \left(= \frac{\Delta e_k + \Delta e_{\text{pot}} + \Delta e_p}{e_A} \right) \quad (45)$$

4.9.1 Volumetric efficiency η_{vol}

$$\eta_{\text{vol}} = \frac{\dot{m}_{\text{real}}}{\dot{m}_{\text{ideal}}} = \frac{\dot{V}_{\text{real}}}{\dot{V}_{\text{ideal}}} \quad (46)$$

Efficiency factor η

4.9.2 Efficiency of a pump-driven system

$$\eta_{\text{pump}} = \frac{P_{\text{hyd}}}{P_{\text{mech}}} = \frac{\dot{m} \cdot Y}{M \cdot \omega} \tag{47}$$

$$\eta_{\text{tot}} = \underbrace{\eta_{\text{el}} \cdot \eta_{\text{mech}} \cdot \eta_{\text{vol}}}_{\text{Pump}} \cdot \eta_{\text{hyd}}^{\text{system}} \tag{48}$$

In the case of an eletrically driven pump, the effective power transferred to the fluid is thus:

$$P_{\text{eff}} = P_{\text{el}} \cdot \eta_{\text{tot}} \iff P_{\text{el}} = \frac{P_{\text{pump}}}{\eta_{\text{pump}}} \tag{49}$$

4.9.3 Efficiency of a turbine-driven system

$$\eta_{\text{turbine}} = \frac{P_{\text{mech}}}{P_{\text{hyd}}} = \eta_{\text{mech}} \cdot \eta_{\text{hyd}} \tag{50}$$

$$\eta_{\text{tot}} = \eta_{\text{turbine}} \cdot \eta_{\text{el}} = \eta_{\text{mech}} \cdot \eta_{\text{hyd}} \cdot \eta_{\text{el}} \tag{51}$$

5 Pipe flows

5.1 Flow characteristics

Reynolds number in pipes

$$Re = \frac{v_m \cdot d}{\nu} \tag{52}$$

Pipe flows

5.1.1 Laminar pipe flow

The pressure loss of a laminar pipe flow is described by the Hagen-Poiseuille:

$$v(r) = \frac{p_1 - p_2}{4\eta \cdot l}\left(R^2 - r^2\right) \tag{53}$$

$$v_m = \frac{v_{\max}}{2} = \frac{p_1 - p_2}{8\eta \cdot l} \cdot R^2$$

$$v_m = \frac{p_1 - p_2}{32\eta \cdot l} \cdot d^2$$

$$\Delta p = 32\eta \cdot v_m \cdot \frac{l}{d^2} \tag{54}$$

5.1.2 Turbolent flow / Pressure lost in pipelines

Flow losses in pipeline systems consist of pressure losses in straight or curved pipes as well as in fittings.

$$\Delta p = \lambda \cdot \frac{l}{d} \cdot \rho \cdot \frac{v_m^2}{2} \tag{55}$$

5.1.3 Resistance coefficient λ

$$\lambda \cdot \frac{l}{d} \cdot \rho \cdot \frac{v_m^2}{2} = 32\eta \cdot v_m \cdot \frac{l}{d^2}$$

$$\lambda = \frac{64\eta}{v_m \cdot d \cdot \rho} = \frac{64}{Re} \tag{56}$$

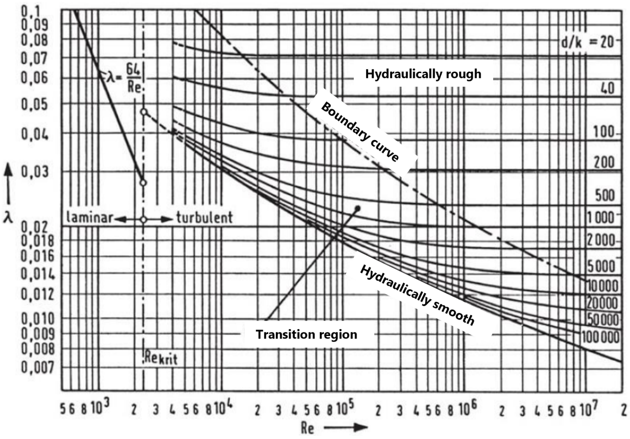
5.1.4 Loss coefficient ζ of a pipe

$$\zeta = \frac{l}{d} \cdot \lambda \tag{57}$$

5.2 Straight pipes

5.2.1 Moody diagram

The resistance coefficient λ depends on the flow characteristics (quantified by the Reynolds number Re) and the relative wall roughness.

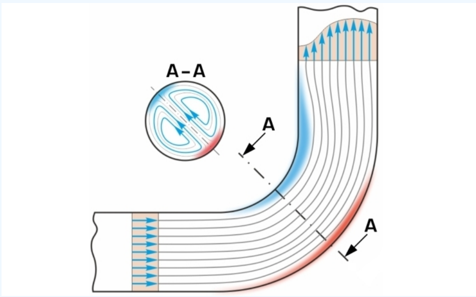


Pipe fittings

In pipeline systems, a portion of the pressure losses is caused by fittings:

$$\Delta p = e_v \cdot \rho = \zeta \cdot \rho \cdot \frac{v_m^2}{2} \tag{58}$$

5.2.2 Elbows



$$\Delta p = \zeta \cdot \rho \cdot \frac{v^2}{2} \tag{59}$$

$$\zeta = f_{Re} \cdot \zeta_u \tag{60}$$

where (given from individual diagrams):

- ζ_u is the geometric resistance coefficient;
- f_{Re} is the Reynolds correction factor.

5.2.3 Diffuser

A diffuser is a section in a pipeline with a continuous increase in cross-sectional area.

The frictional losses Δp_v in a diffuser are given by:

$$\Delta p_v = \frac{\zeta \rho v_1^2}{2} \tag{61}$$

$$\Delta p_{v,\text{ideal}} = \rho \frac{v_2^2 - v_1^2}{2} \tag{62}$$

Pipe fittings

The diffuser efficiency η_D according to Bernoulli:

$$\eta_D = \frac{p_2 - p_1}{\Delta p_B} = 1 - \zeta \frac{1}{1 - \left(\frac{A_1}{A_2}\right)^2} = \frac{c_p}{c_{p,\text{ideal}}} \quad (63)$$

The various coefficients are stated as:

$$c_p = \frac{2\,(p_2 - p_1)}{\rho v_1^2} = \eta_D \cdot c_{p,\text{ideal}} \quad (64)$$

$$c_{p,\text{ideal}} = 1 - \left(\frac{A_1}{A_2}\right)^2 \quad (65)$$

$$\zeta_1 = c_{p,\text{ideal}} - c_p \quad (66)$$

The opening angle of a diffuser can be calculated as:

$$\tan(\theta) = \frac{d_2 - d_1}{2L} \quad (67)$$

$$\varphi = 2\theta \quad (68)$$

The optimal angle φ_{opt} is between 6-20 degrees.

5.2.4 Inlets

When a stationary fluid is introduced from a large container into a pipe, losses occur due to acceleration and the formation of separation bubbles.

- sharp edge: $0.45 < \zeta < 0.50$
- broken edge: $\zeta = 0.20$

5.2.5 Outlets

For an outlet into a large basin, the entire kinetic energy is converted into static pressure loss, meaning $\zeta = 1$ can be assumed.

5.2.6 Valves and fittings

ζ-values are specified by different manufacturer.

6 Linear momentum theorem

Linear momentum

$$\vec{I} = m \cdot \vec{v}\,[Ns] \quad (69)$$

6.1 Linear momentum balance

Momentum flux

The change in motion is a change in linear momentum over time:

$$\vec{F}_{\text{res}} = \frac{d\vec{I}}{dt} = \dot{\vec{I}} = \frac{d\,(m \cdot \vec{v})}{dt} \quad (70)$$

at constant mass:

$$\dot{\vec{I}} = m \cdot \vec{a} = \dot{m} \cdot \vec{v} \quad (71)$$

6.2 System of forces

Force balance

The sum of all external forces on a control volume is the difference of momentum flow:

$$\vec{F}_{\text{res}} = \sum_i \vec{F}_i = \dot{\vec{I}}_{\text{out}} - \dot{\vec{I}}_{\text{in}} \quad (72)$$

expanded to:

$$\dot{\vec{I}}_{\text{out}} - \dot{\vec{I}}_{\text{in}} = \dot{m}\,(\vec{v}_2 - \vec{v}_1) = \rho \cdot \dot{V} \cdot (\vec{v}_2 - \vec{v}_1) \quad (73)$$

where:

$$m_1 = \rho_1 \cdot \dot{V}_1 \cdot \Delta t = \rho_1 \cdot v_1 \cdot A_1 \cdot \Delta t$$

$$m_2 = \rho_2 \cdot \dot{V}_2 \cdot \Delta t = \rho_2 \cdot v_2 \cdot A_2 \cdot \Delta t \quad (74)$$

$$\vec{I}_{\text{in}} = \vec{I}_1 = m_1 \cdot \vec{v}_1 = \rho_1 \cdot v_1 \cdot A_1 \cdot \Delta t \cdot \vec{v}_1$$

$$\vec{I}_{\text{out}} = \vec{I}_2 = m_2 \cdot \vec{v}_2 = \rho_2 \cdot v_2 \cdot A_2 \cdot \Delta t \cdot \vec{v}_2 \quad (75)$$

6.2.1 Momentum as vector

$$\dot{\vec{I}}_{xyz} = \left(\dot{I}_x \dot{I}_y \dot{I}_z\right)$$

$$\dot{I} = \sqrt{\dot{I}_x^2 + \dot{I}_y^2 + \dot{I}_z^2} \quad (76)$$

The external forces can consist of pressure forces F_P, body forces (support forces) F_B, gravitational forces F_G, and frictional forces F_F:

$$\vec{F}_{\text{res}} = \sum_i \vec{F}_i = \vec{F}_P + \vec{F}_B + \vec{F}_G + \vec{F}_F \quad (77)$$

6.3 Control volume

Linear momentum calculation steps

I Step: Select a suitable **coordinate system** and draw it in a sketch of the flow problem;

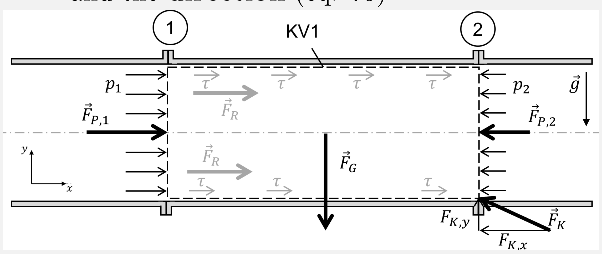
II Step: Select **control volume** sensibly and draw it in the sketch. The balance boundary should be set so that the external forces on its surface are knwon;

III Step: Draw in the **forces** $\vec{F}_P, \vec{F}_B, \vec{F}_G, \vec{F}_F$ acting on the CV from the outside and calculate them from the known quantities.

IV Step: Calculate the **linear momentum fluxes** at the outlet and inlet and insert them as the resulting force (eq. 72)

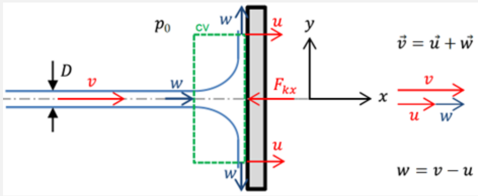
V Step: Dissolve the momentum balance equations according to the **sought quantity** or its components and calculate them.

VI Step: If necessary, calculate the **magnitude** and the **direction** (eq. 75)



Linear momentum calculation steps

6.3.1 Plates



$$F_{kx} = \dot{m} \cdot v = \rho A v^2 \tag{78}$$

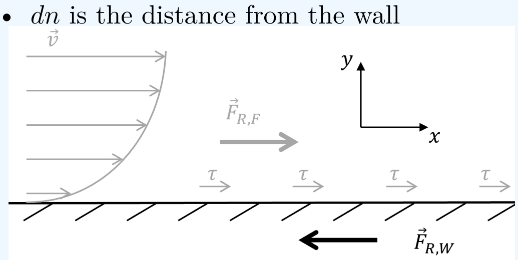
6.4 Momentum on x-direction

Wall shear stress

The shear stress τ_w is the force per unit area acting on the pipe's walls:

$$\tau_w = \frac{dv}{dn} \cdot \eta \tag{79}$$

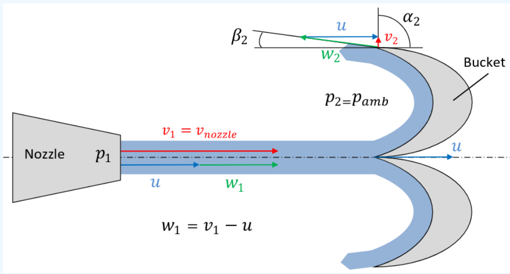
where:
- dv is the velocity difference;
- dn is the distance from the wall



$$|F_{K,x}| = |A\,(p_{\text{in}} - p_{\text{out}})| = |\tau_w \cdot A \cdot l| \tag{80}$$

6.5 Pelton turbine

Pelton bucket



6.5.1 Velocities

- v_1: absolute velocity [m/s]
- u: radial velocity [m/s]
- w_1: relative velocity (turbine POV) [m/s]

$$v_{\text{nozzle}} = \sqrt{2g\Delta h} = \sqrt{\frac{2\Delta p}{\rho}}$$

$$w_1 = v_{\text{nozzle}} - u = \sqrt{\frac{2\Delta p}{\rho}} - D_{\text{wheel}} \cdot \pi \cdot n_{\text{wheel}} \tag{81}$$

$$u = \frac{v_{\text{nozzle}}}{2} \tag{82}$$

Pelton bucket

6.5.2 Rotational speed

$$k_u = \frac{u}{v_{\text{nozzle}}} \tag{83}$$

6.5.3 Hydraulic power

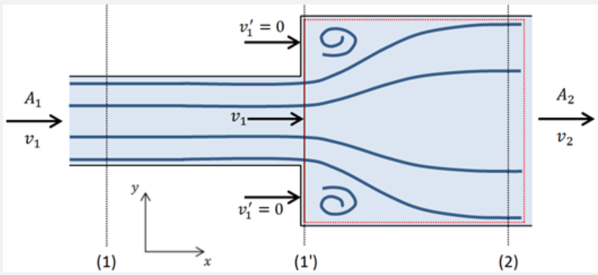
$$P_{\text{hyd}} \approx \rho \cdot g \cdot H \cdot \dot{V} \approx \Delta p \cdot \dot{V}\,[W] \tag{84}$$

Laminar pipe velocity (section 5)

$$v(r) = \frac{\Delta p \cdot R^2}{4\eta \cdot l}\left(1 - \left(\frac{r}{R}\right)^2\right) \tag{85}$$

6.6 Borda-Carnot diffuser

Borda-Carnot diffuser



The continuity equation (eq 12, 13, 14) can be applied for the pipe expansion.

6.6.1 Pressure difference

$$\sum F_x = p_1 A_1 - p_2 A_2 = \dot{m}\,(v_2 - v_1) \tag{86}$$

$$p_1 - p_2 = \rho \cdot v_2 \cdot (v_2 - v_1) = \rho \cdot v_2^2 \left(1 - \frac{v_1}{v_2}\right)$$

$$= \rho \cdot v_2^2 \cdot \left(1 - \frac{A_1}{A_2}\right) = \rho \cdot v_1^2 \cdot \frac{A_1^2}{A_2^2} \cdot \left(1 - \frac{A_2}{A_1}\right)$$

$$p_1 - p_2 = \rho \cdot v_1^2 \cdot \frac{A_1}{A_2} \cdot \left(\frac{A_1}{A_2} - 1\right) \tag{87}$$

6.6.2 Maximum pressure

The maximum possible pressure increase can be archieved with an area ratio of $A_1/A_2 = 0.5$. Thus:

$$(p_2 - p_1)_{\max} = \frac{\rho \cdot v_1^2}{4} \tag{88}$$

6.6.3 Pressure loss in ideal diffusers

$$\Delta p_{V,\text{id}} = \frac{\rho}{2}\,(v_1 - v_2)^2 = \frac{\rho \cdot v_1^2}{2}\left(1 - \frac{A_1^2}{A_2^2}\right) \tag{89}$$

6.6.4 Pressure loss in real diffusers

$$\Delta p_V = \Delta p_{V,\text{id}} - \zeta \frac{\rho \cdot v_m^2}{2} = \frac{\rho \cdot v_1^2}{2}\left(1 - \frac{A_1^2}{A_2^2} - \zeta\right) \tag{90}$$

Borda-Carnot diffuser

6.6.5 Flow losses

$$\frac{p_1}{\rho} + \frac{v_1^2}{2} = \frac{p_2}{\rho} + \frac{v_2^2}{2} + e_V$$

$$e_V = \frac{v_1^2}{2} \cdot \left(\frac{A_1}{A_2} - 1\right)^2 = \frac{v_1^2}{2} \cdot \zeta \tag{91}$$

Hence:

$$\zeta = \left(\frac{A_1}{A_2} - 1\right)^2 \tag{92}$$

6.7 Analyze of momentum equation

Momentum equation

for $A_2 > A_1 \implies p_2 > p_1$; $\quad 0 < \zeta < 1$

for $A_2 = A_1 \implies p_2 = p_1$; $\quad \zeta = 0$

for $A_2 \to \infty \implies p_2 = p_1$; $\quad \zeta = 1$

for $A_2 = 2A_1 \implies p_2 - p_1$ becomes maximal

Assuming $r = A_2/A_1$, we know $\zeta = (1-r)^2$:

if $r = 0.5 \implies \zeta = 0.5$

if $r = 0.25 \implies \zeta = 0.5625$

if $r = 0.75 \implies \zeta = 0.0625$

7 Angular momentum theorem

Angular momentum equation

7.1 Moment of inertia

Considering the moment of inertia as a scalar quantity of a point mass instead of a tensor:

$$J_{\text{PM}} = r^2 \cdot m = \int_m r^2 \, dm \, [kg \; m^2] \tag{93}$$

7.1.1 Angular momentum D

The angular momentum D of a mass m is rotating around a point O with an angular velocity ω is:

$$D = m \cdot v \cdot r = m \cdot r^2 \cdot \omega \, [Nm \cdot s] \tag{94}$$

7.2 Angular momentum flux balance

The sum of all ext. torques on a CV is equal to the difference of the in/out angular momentum flux:

$$\vec{M}_{\text{res}} = \sum_i \vec{M}_i = \dot{\vec{D}}_{\text{out}} - \dot{\vec{D}}_{\text{in}} \tag{95}$$

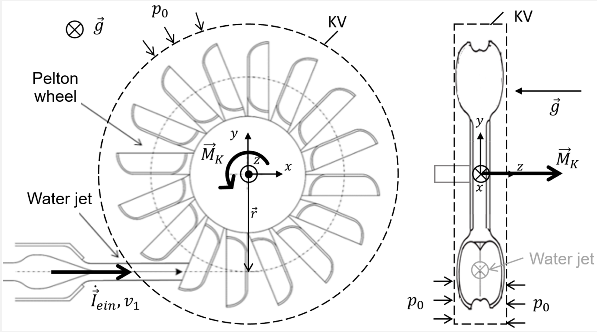
$$\dot{\vec{D}} = \vec{r} \times \dot{\vec{I}} = \vec{r} \times (\dot{m} \cdot \vec{v}) \tag{96}$$

7.2.1 Angular momentum as vector

As (eq. 75), the angular momentum is a vector.

7.3 Angular momentum application

Pelton turbine



I Step: xy-coordinates and CV

II Step: Relevant forces $\vec{F}_P, \vec{F}_G, \vec{F}_F, \vec{F}_B$

$$M_{\text{res,z}} = M_{B,z} \tag{97}$$

III Step: Angular momentum flux calculations:

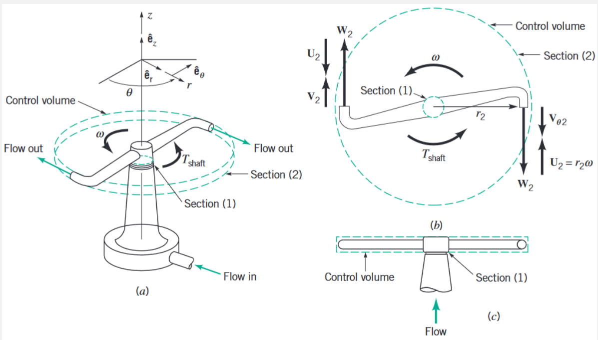
$$\dot{D}_{\text{out},z} - \dot{D}_{\text{in},z} = M_{B,z} = -r \cdot \dot{m} \cdot v_1 \tag{98}$$

7.3.1 Delivered power

$$P = \omega \cdot M_{B,z} = \underbrace{r \cdot \frac{v_1}{2}}_{r \cdot u} \cdot \underbrace{r \cdot \dot{m} \cdot v_1}_{M_{B,z}}$$

$$= r^2 \cdot \dot{m} \cdot \frac{v_1^2}{2} = r^2 \cdot \rho \cdot \frac{v_1^3}{2} \cdot A_{\text{jet}} \tag{99}$$

Lawn sprinkler



7.3.2 Tangential (flat) jet exit

$$\dot{\vec{D}}_{\text{out},z} - \dot{\vec{D}}_{\text{in},z} = M_{B,z}$$

$$\dot{\vec{D}}_{\text{out},z} = -2r \cdot \dot{m} \cdot v_1 = -2r \cdot \dot{m} \cdot (w_u - u) \tag{100}$$

$$\dot{\vec{D}}_{\text{in},z} = 0 \tag{101}$$

7.3.3 Deflection α in plane, tilt β out of plane

$$w_{xy} = \cos\beta w_1, \quad w_z = \sin\beta w_1 \tag{102}$$

$$w_u = \cos\alpha w_{xy} = \cos\alpha \cos\beta w_1 \quad v_u = w_u - u \tag{103}$$

Translation	**Rotation**
Location: $\vec{x}$ [m]	Angle: $\vec{\varphi}$ [° or rad]
Velocity: $= \vec{v} = \frac{d\vec{x}}{dt}$ [m/s]	Angular velocity: $\vec{\omega} = \frac{d\varphi}{dt}$ [1/s]
Mass: m [kg]	Mass inertia $J = \int_m r^2 \, dm$ [kg m^2]
Linear momentum: $\vec{I} = m \cdot \vec{v}$ [kg m/s]	Angular momentum $\vec{D} = J \cdot \vec{\omega} = \vec{r} \times \vec{I}$ [kg m^2/s]
Linear momentum flux: $\dot{\vec{I}} = \frac{d\vec{I}}{dt}$ [N]	Angular momentum flux: $\dot{\vec{D}} = \frac{d\vec{D}}{dt} = \vec{r} \times \dot{\vec{I}}$ [N m]
Force: $\vec{F} = \dot{\vec{I}}$ [N]	torque: $\vec{M} = \vec{r} \times \vec{F} = \dot{\vec{D}}$ [N m]
Linear momentum equation: $\sum \vec{F} = \dot{\vec{I}}_{off} - \dot{\vec{I}}_{on}$	Angular momentum equation: $\sum \vec{M} = \dot{\vec{D}}_{off} - \dot{\vec{D}}_{on}$