Mathematics 2A
HSLU, Semester 2

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Part I

Differential Equations Theory

1 Introduction

A **differential equation** is an equation in which derivatives of an unknown function appear. For example, consider the simple differential equation Remark: This equation asserts that the instantaneous rate of change

$$\boxed{\frac{dH}{dG} = H}$$

of H with respect to G equals H itself. Its general solution is $H(G) = \mathcal{C}e^G$ with $\mathcal{C}$ an arbitrary constant.

2 Separation of Variables

For a separable differential equation of the form

$$\frac{dy}{dx} = f(x)\,g(y),$$

we rewrite it as

$$\frac{dy}{g(y)} = f(x)\,dx.$$

Remark: After integration, one typically obtains an implicit solution that can be solved (if possible) for y.

$$\boxed{\int \frac{dy}{g(y)} = \int f(x)\,dx}$$

Warning: Ensure that $g(y) \neq 0$ on the interval of interest.

3 Linear Differential Equations

A first-order linear differential equation can be written in the standard form

$$y' + p(x)y = q(x).$$

Its general solution is given by

$$y = y_h + y_p,$$

where y_h is the general solution of the homogeneous part

$$y' + p(x)y = 0,$$

and y_p is any particular solution of the full inhomogeneous equation. Remark: The principle of superposition

$$\boxed{y_h = A\exp\left(-\int p(x)dx\right)}$$

applies to the homogeneous equation; that is, any linear combination of solutions is again a solution.

4 Exponential Growth and Decay

Many natural processes obey the simple law

$$\frac{dP}{dt} = kP.$$

Its general solution is

$$P(t) = P(0)e^{kt}.$$

Remark: This model applies not only to population growth but also to radioactive decay (with $k < 0$).

$$\boxed{P(t) = P_0 e^{kt}}$$

5 Graphical Representation: Slope Fields

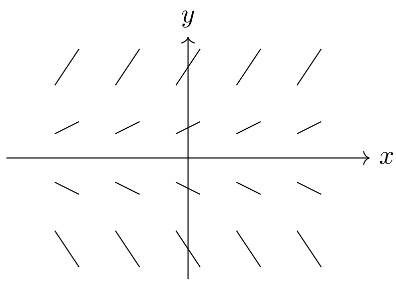
A slope field (or direction field) helps visualize the behavior of solutions of a differential equation by drawing, at selected points (x, y), short line segments whose slope is given by the value of $f(x, y)$ in

$$y' = f(x, y).$$

For example, for the differential equation

$$y' = y,$$

the slope at each point is simply the y-value. The following TikZ figure illustrates a portion of this slope field.



Remark: For $y' = y$, the slope at each point equals its y-coordinate. Thus, solution curves such as $y = Ce^x$ naturally emerge from the field.

Part II

Mathematical Formulary

6 Lines and Linear Functions

6.1 Slope and Equation of a Line

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$$

$$\boxed{y - y_1 = m(x - x_1)}$$

$$\boxed{y = mx + b}$$

Remark: These formulas describe the fundamental properties of straight lines in the Cartesian plane.

7 Exponents and Logarithms

7.1 Working with Exponents

$$\boxed{a^x \cdot a^t = a^{x+t}}$$

$$\boxed{\frac{a^x}{a^t} = a^{x-t}}$$

$$\boxed{(a^x)^t = a^{xt}}$$

$$\boxed{y = \ln x \Longleftrightarrow e^y = x}$$

7.2 Definition of the Natural Logarithm

Remark: For instance, $\ln 1 = 0$ because $e^0 = 1$.

7.3 Logarithmic Identities

$$\boxed{\ln(AB) = \ln A + \ln B}$$

$$\boxed{\ln\left(\frac{A}{B}\right) = \ln A - \ln B}$$

$$\boxed{\ln A^p = p \ln A}$$

8 Distances and Midpoint Formulas

8.1 Distance Formula

$$\boxed{D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

8.2 Midpoint Formula

$$\boxed{\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)}$$

9 Quadratic Equations

$$\boxed{ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}$$

10 Factoring Special Polynomials

$$\boxed{x^2 - y^2 = (x+y)(x-y)}$$

$$\boxed{x^3 + y^3 = (x+y)(x^2 - xy + y^2)}$$

$$\boxed{x^3 - y^3 = (x-y)(x^2 + xy + y^2)}$$

11 Conic Sections

11.1 Circles

$$\boxed{(x-h)^2 + (y-k)^2 = r^2}$$

11.2 Ellipses

11.3 Hyperbolas

Remark: The asymptotes of a hyperbola are given by $y = \pm\frac{b}{a}x$.

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

12 Geometric Formulas

12.1 Conversion Between Radians and Degrees

$$\boxed{\pi \text{ radians} = 180^\circ}$$

12.2 Circle Geometry

$$\boxed{A = \pi r^2, \quad C = 2\pi r}$$

12.3 Sector of a Circle

$$\boxed{A = \frac{1}{2} r^2 \vartheta, \quad s = r\vartheta \ \vartheta \text{ in radians.}}$$

12.4 Volumes and Surface Areas of Solids

- Sphere: $V = \frac{4}{3}\pi r^3, \quad A = 4\pi r^2$.
- Cylinder: $V = \pi r^2 h$.
- Cone: $V = \frac{1}{3}\pi r^2 h$.

13 Trigonometric Functions and Identities

13.1 Definitions

For a right triangle with hypotenuse r and legs x and y:

$$\boxed{\sin\vartheta = \frac{y}{r}, \quad \cos\vartheta = \frac{x}{r}, \quad \tan\vartheta = \frac{y}{x}}$$

13.2 Fundamental Identity

$$\boxed{\sin^2\vartheta + \cos^2\vartheta = 1}$$

13.3 Angle Sum and Difference Formulas

$$\boxed{\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B}$$

$$\boxed{\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B}$$

13.4 Double Angle Formulas

$$\boxed{\sin(2A) = 2 \sin A \cos A}$$

$$\boxed{\cos(2A) = 2\cos^2 A - 1 = 1 - 2\sin^2 A}$$

14 Binomial Expansions

The binomial expansion for $(x+y)^n$ is given by Remark: For $(x-y)^n$, the signs alternate accordingly.

$$\boxed{(x+y)^n = x^n + n\,x^{n-1}y + \tfrac{n(n-1)}{2!}x^{n-2}y^2 + \cdots + y^n}$$

15 Differentiation Rules

1. $(f(x) \pm g(x))' = f'(x) \pm g'(x)$.
2. $(k\,f(x))' = k\,f'(x)$.
3. $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.
4. $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$.
5. $(f(g(x)))' = f'(g(x)) \cdot g'(x)$.
6. $\frac{d}{dx}(x^n) = nx^{n-1}$.
7. $\frac{d}{dx}(e^x) = e^x$.
8. $\frac{d}{dx}(a^x) = a^x \ln a, \quad a > 0$.
9. $\frac{d}{dx}(\ln x) = \frac{1}{x}$.
10. $\frac{d}{dx}(\sin x) = \cos x$.
11. $\frac{d}{dx}(\cos x) = -\sin x$.
12. $\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x}$.
13. $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$.
14. $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$.

16 Integration Rules

1. $\displaystyle\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$.
2. $\displaystyle\int k\,f(x)\,dx = k\int f(x)\,dx$.
3. $\displaystyle\int f(g(x))g'(x)\,dx = \int f(w)\,dw, \quad w = g(x)$.
4. $\displaystyle\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$.

17 Taylor Series Expansions

The Taylor series of $f(x)$ about $x = a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots .$$

Important examples include:

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \\
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots \quad (|x| < 1), \\
(1+x)^p &= 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots.
\end{aligned}
$$

18 Complex Numbers and Euler's Formula

A complex number z is written as

$$z = x + yj, \quad x, y \in \mathbb{R}.$$

Its magnitude is

$$|z| = \sqrt{x^2 + y^2},$$

and its conjugate is

$$\bar{z} = x - yj.$$

Euler's formula states that

$$e^{jt} = \cos t + j \sin t,$$

so any complex number can be written in polar form as

$$z = re^{j\varphi}, \quad r \geq 0, \ -\pi < \varphi \leq \pi.$$