Mathematics 3A
HSLU, Semester 3

Matteo Frongillo

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Contents

Part I

Planes and surfaces in space

1 Plane π in space

Let π denote the plane:

$$s_x \in \pi, s_y \in \pi, s_z \in \pi$$

$$\pi : ax + by + cz + d = 0$$

For $S_x \in \pi \implies 1a + 0b + 0c + d = 0$, hence
$a + d = 0$

For $S_y \in \pi \implies 0a + 2b + 0c + d = 0$, hence
$2b + d = 0$

for $S_z \in \pi \implies 0a + 0b + 3c + d = 0$, hence
$3c + d = 0$

$$\begin{cases} a + d = 0 \\ 2b + d = 0 \\ 3c + d = 0 \end{cases} \implies \begin{cases} a = -d \\ 2b = -d \\ 3c = -d \end{cases}$$

Case 1:

$$d = 0 \implies a = 0, b = 0, c = 0 \implies \pi : 0 = 0 \implies \text{NOT a plane!}$$

Case 2:

$$d \neq 0 \implies \pi : \frac{ax + by + cz + d}{d} = 0 \implies \frac{a}{d}x + \frac{b}{d}y + \frac{c}{d}z + 1 = 0$$

Hence:

$$\begin{cases} a = -d \\ 2b = -d \\ 3c = -d \end{cases} \implies \begin{cases} \frac{a}{d} = -1 \\ \frac{b}{d} = -\frac{1}{2} \\ \frac{c}{d} = -\frac{1}{3} \end{cases}$$

Which leads to:

$$\pi : -x - \frac{1}{2}y - \frac{1}{3}z + 1 = 0$$

Remark: the equation of a plane is defined up to a multiplication by a real number different from 0

e.g.: the same planed is shared between those 3 equations
ex 1)

$$z = 0 \iff 5z = 0 \iff -10z = 0$$

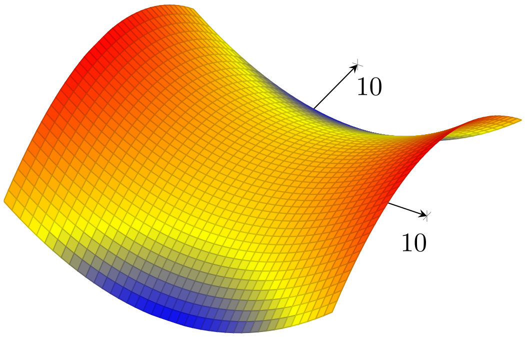
ex 2)

$$-x - \frac{1}{2}y - \frac{1}{3}z + 1 = 0 \iff 6x + 3y + 2z + 6 = 0$$

2 Functions in two variables x and y

Let us take $\pi : x^2 - y^2 = 0$ as example.

The plot would look like this:



2.1 Spheres

3 Linear functions of two variables

We say that z is a *linear function* of x and y, if there are constant a, b and d such that:

$$\boxed{z = ax + by + d}$$

holds. Alternatively: if there are constant A, B, C, D, with $C \neq 0$, such that:

$$\boxed{Ax + By + Cz + D = 0}$$

holds. Since $C \neq 0$, we can rearrange this equation into:

$$\boxed{z = -\frac{Ax}{C} - \frac{By}{C} - \frac{D}{C}}$$

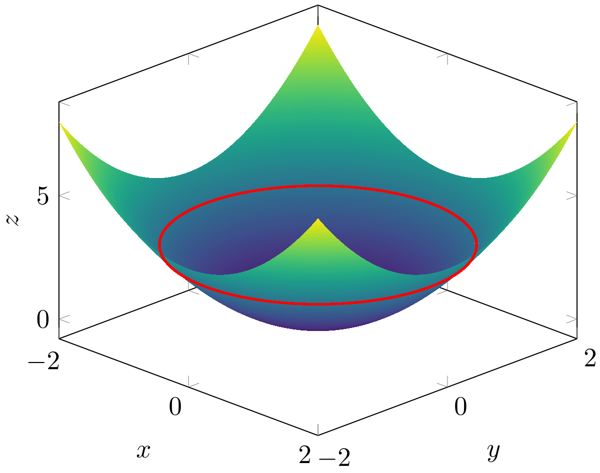
4 Contour lines

$$\begin{cases} z = f(x, y) \\ z = k \qquad k \in \mathbb{R} \end{cases}$$

$z = k$ represents all the possible horizontal planes

Ex:

$$\begin{cases} z = x^2 - y^2 \\ z = k \end{cases} \implies \begin{cases} k = x^2 - y^2 \\ z = k \end{cases}$$



All the planes with equation $z = k$ are parallel to the coordinate planes $z = 0$.

When $z = k = 0$, the circle is reduced to a point, the origin.

When $k < 0$, the equation $x^2 + y^2 = k$ has no solution in $\mathbb{R}$.

When $k > 0$, the equation $x^2 + y^2 = k$ represents a circle with radius $\sqrt{k}$ centered at the origin.

5 Cylinders

A cylinder is a surface generated by all the lines parallel to a given line d and passing through a given curve $\mathcal{C}$.

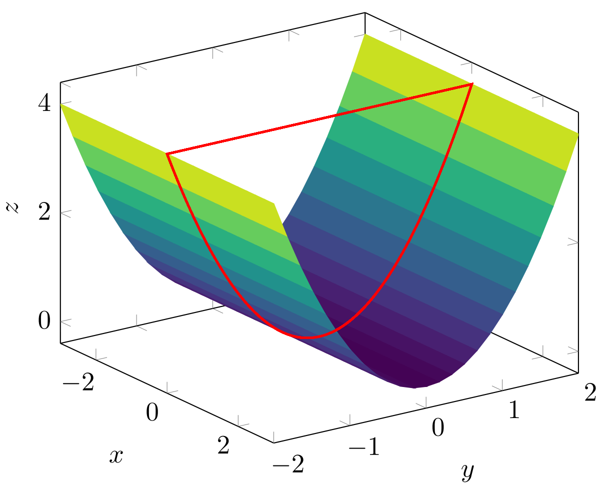
5.1 Property

Whenever you have a polynomial equation of degree at least 2 with a missing variable, then you have a cylinder (up to few exceptions).

Ex:

$$z = y^2 \implies y^2 - z = 0$$

This is a cylinder with generatrix parallel to the x axis and directrix the parabola $y^2 - z = 0$ in the yz plane.



Part II

Partial derivatives

For a multivariable function $f(x, y, ...)$, the partial derivative to one variable measures the instantaneous rate of change of f when that variable changes and the others are held constant:

$$\boxed{\frac{\partial z}{\partial x} = f_x(x, y)}$$

If z is a function of x and y, we define:

The rate of change of z with respect to x, with y fixed, at the point (x, y)=(a, b) as

$$\boxed{\frac{\partial z}{\partial x}\Big|_{(x,y)=(a,b)} = \lim_{h\to 0} \frac{Z\big|_{(x,y)=(a+h,b)} - Z\big|_{(x,y)=(a,b)}}{h}}$$

The rate of change of z with respect to y, with x fixed, at the point $(x, y) = (a, b)$ as

$$\boxed{\frac{\partial z}{\partial x}\Big|_{(x,y)=(a,b)} = \lim_{h\to 0} \frac{Z\big|_{(x,y)=(a,b+h)} - Z\big|_{(x,y)=(a,b)}}{h}}$$

For the lectures, we will be using the formula with 2-steps difference ($\Delta z_a = (a + h, b) - (a - h, b)$):

$$\boxed{\begin{aligned} \frac{\partial z}{\partial x}\Big|_{(x,y)=(a,b)} &= \frac{Z\big|_{(x,y)=(a+h,b)} - Z\big|_{(x,y)=(a-h,b)}}{2h} \\ \frac{\partial z}{\partial y}\Big|_{(x,y)=(a,b)} &= \frac{Z\big|_{(x,y)=(a,b+h)} - Z\big|_{(x,y)=(a,b-h)}}{2h} \end{aligned}}$$

6 Local linearization

6.1 Tangent plane of a function at point P

Let $f(x, y)$ be our function and $P(a, b)$ a point, $P \in f$:

$$\boxed{f(x, y) \approx f(a, b) + \frac{\partial}{\partial x} f(a, b)(x - a) + \frac{\partial}{\partial y} f(a, b)(y - b)}$$

7 Gradient

The gradient of a function $z = f(x, y)$ is defined by:

$$\boxed{\begin{aligned} &\text{grad } f = \nabla f = f_x \overrightarrow{e_x} + f_y \overrightarrow{e_y} = \begin{pmatrix} f_x \\ f_y \end{pmatrix} \\ &\text{where } f_x = \frac{\partial f}{\partial x} \text{ and } f_y = \frac{\partial f}{\partial y} \end{aligned}}$$

7.1 Geometrical properties of the gradient vector ∇ in the plane

If f is differentiable at the point (a, b) and $\nabla f \neq \overrightarrow{0}$, then the following holds:

$\nabla \mathbf{f}(\mathbf{a}, \mathbf{b})$:

- is perpendicular to the contour line of f through (a, b)
- points in the direction of the maximum rate of change f

The length $\|\nabla \mathbf{f}(\mathbf{a}, \mathbf{b})\|$ of the gradient vector is:

- the maximum rate of change f at this point
- large when the contour lines are close together
- small when the contour lines are far apart

7.2 Gradient of a function of three variables

The gradient of a function $w = f(x, y, z)$ is defined by:

$$\text{grad } f = \nabla f = f_x \overrightarrow{e_x} + f_y \overrightarrow{e_y} + f_z \overrightarrow{e_z} = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

where $f_x = \dfrac{\partial f}{\partial x}$, $f_y = \dfrac{\partial f}{\partial y}$, and $f_z = \dfrac{\partial f}{\partial z}$

7.3 Second-order partial derivatives of $z = f(x, y)$

A function $z = f(x, y)$ has two first-order partial derivatives, f_x and f_y, and four second-order partial derivatives:

1. $\dfrac{\partial^2 z}{\partial x^2} = f_{xx}(x, y) = (f_x)_x(x, y)$,
2. $\dfrac{\partial^2 z}{\partial x \partial y} = f_{yx}(x, y) = (f_y)_x(x, y)$,
3. $\dfrac{\partial^2 z}{\partial y \partial x} = f_{xy}(x, y) = (f_x)_y(x, y)$,
4. $\dfrac{\partial^2 z}{\partial y^2} = f_{yy}(x, y) = (f_y)_y(x, y)$

Usually, parenthesis are omitted, writing directly f_{xy} instead of $(f_x)_y$, and $\dfrac{\partial^2 z}{\partial y \partial x}$ instead of $\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right)$.

7.4 Equality of mixed partial derivatives (Schwarz's Theorem)

If f_{xy} and f_{yx} are continuous at a point (a, b) inside the domain, then:

$$\boxed{f_{xy}(a, b) = f_{yx}(a, b)}$$

8 Directional derivatives in the plane

8.1 Directional derivative of f at $P(a,b)$ in the direction of $\overrightarrow{u}$

If $\overrightarrow{e_u} = \overrightarrow{u} = u_1\overrightarrow{e_x} + u_2\overrightarrow{e_y}$ is a unit vector $\|u\| = 1$, we define the directional derivative $\dfrac{\partial f}{\partial \overrightarrow{u}} = f_{\overrightarrow{u}}$ by

$$\boxed{\frac{\partial f}{\partial \overrightarrow{u}}(a,b) = f_{\overrightarrow{u}}(a,b) = \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}}$$

8.2 Gradient and directional derivative

If f is differentiable and $\overrightarrow{e_u} = u_1\overrightarrow{e_x} + u_2\overrightarrow{e_y}$ is the unit vector in the direction of $\overrightarrow{u}$, then:

$$\boxed{\frac{\partial f}{\partial \overrightarrow{u}}(a,b) = f_{\overrightarrow{u}}(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2 = \nabla f(a,b) \cdot \overrightarrow{e_u}}$$

9 Critical points

9.1 Discriminant

Let (x_0, y_0) be a critical point. Furthermore, let

$$\boxed{D(x_0,y_0) = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - \big(f_{xy}(x_0,y_0)\big)^2}$$

Then the following holds:

- If $D > 0$ and $f_{xx} > 0$, then f has a local minimum at (x_0, y_0)
- If $D > 0$ and $f_{xx} < 0$, then f has a local maximum at (x_0, y_0)
- If $D < 0$, then f has a saddle point at (x_0, y_0)
- If $D = 0$, no conclusion can be made

10 Constraints and Lagrange Multipliers

10.1 Lagrange multiplier λ

The scalar λ measures how sensitive the optimal value of f is with respect to small changes in the constraint level c. Formally,

$$\boxed{\lambda = \frac{\partial f^*}{\partial c}}$$

where f^* denotes the optimal value of f. A positive λ indicates that relaxing the constraint (c larger) increases the optimal value of f.

10.2 Graphical representation

The optimization of $f(x,y)$ under the constraint $g(x,y) = c$ can be visualized as searching for points where a level curve of f is tangent to the constraint curve. At an optimum, the gradients are parallel:

$$\boxed{\nabla f(x,y) = \lambda\, \nabla g(x,y)}$$

10.3 Lagrange function $\mathcal{L}$

When optimizing $f(x, y)$ under the constraint $g(x, y) = c$, the Lagrange function is used:

$$\boxed{\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c)}$$

The partial derivatives must be calculated:

$$\boxed{\begin{aligned}\frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} \\ \frac{\partial \mathcal{L}}{\partial y} &= \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= -\big(g(x, y) - c\big)\end{aligned}}$$

The stationary points of $\mathcal{L}$ satisfy:

$$\boxed{\begin{aligned}&\frac{\partial \mathcal{L}}{\partial x} = 0 \\ &\frac{\partial \mathcal{L}}{\partial y} = 0 \\ &g(x, y) = c\end{aligned}}$$

Solutions (x, y, λ) of this system give the candidate extrema of f under the constraint $g(x, y) = c$.

Part III

Integration of functions with multiple variables

11 Domain of integration Ω

Let $f : \Omega \subset \mathbb{R}^2 \to R$. The set Ω is a region in the xy-plane over which the double integral

$$\boxed{\iint_\Omega f(x,y)\ dx\ dy}$$

is taken.

12 Double integrals as iterated integrals

If the region R is a rectangle with $a \leq x \leq b$ and $c \leq y \leq d$ and if f is continuous in the region R, then the integral of f over R is equal to the iterated integral

$$\boxed{\int\limits_R f\, dA = \int\limits_{y=c}^{d} \int\limits_{x=a}^{b} f(x,y)\, dx\, dy}$$

The iterated integrals can also be written as

$$\boxed{\int\limits_c^d \int\limits_a^b f(x,y)\, dx\, dy}$$

12.1 Double integral over rectangles

$$\boxed{\int\limits_R f(x,y)\, dA = \int\limits_c^d \int\limits_a^b f(x,y)\, dx\, dy = \int\limits_a^b \int\limits_c^d f(x,y)\, dy\, dx}$$

12.2 Triangular regions

For the triangle with vertices $(0,0)$, $(1,0)$, $(0,1) \implies 0 \leq y \leq 1,\ 0 \leq x \leq 1 - y$:

$$\boxed{\int\limits_{y=0}^{1} \int\limits_{x=0}^{1-y} f(x,y)\, dx\, dy}$$

Equivalently $0 \leq x \leq 1,\ 0 \leq y \leq 1 - x$:

$$\boxed{\int\limits_{x=0}^{1} \int\limits_{y=0}^{1-x} f(x,y)\, dy\, dx}$$

12.3 Double integral over general regions

If the region Ω is not a rectangle, one must describe it using variable limits that follow the boundary of Ω

12.3.1 x-simple region

If the region $\Omega = \{(x,y) \mid a \leq x \leq b,\ \varphi_1(x) \leq y \leq \varphi_2(x)\}$, then

$$\boxed{\int\limits_a^b \int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\, dy\, dx}$$

12.3.2 y-simple region

If the region $\Omega = \{(x,y) \mid c \leq y \leq d,\ \psi_1(y) \leq x \leq \psi_2(y)\}$, then

$$\boxed{\int\limits_c^d \int\limits_{\psi_1(y)}^{\psi_2(y)} f(x,y)\, dx\, dy}$$

12.4 Double integrals in Polar coordiates

12.4.1 Polar coordinates

Polar coordinates are defined as the coordinate change

$$\boxed{f : (0, +\infty) \times (-\pi, \pi] \to \mathbb{R}^2 \setminus \left\{(x,0) \in \mathbb{R}^2 \mid x \leq 0\right\}}$$

given by

$$\boxed{f(r,\varphi) = (r\cos\varphi, r\sin\varphi)}$$

12.4.2 Integration formula

To compute an integral in polar coordinates:

$$\boxed{\begin{array}{l} x = r\cos\varphi, \\ y = r\sin\varphi, \\ x^2 + y^2 = r^2 \end{array}}$$

and

$$\boxed{dA = r\ \mathrm{d}\varphi\ \mathrm{d}r \quad \text{and} \quad dA = r\ \mathrm{d}r\ \mathrm{d}\varphi}$$

13 Triple integrals as iterated integrals

If the region V is a box with $a \leq x \leq b$, $c \leq y \leq d$, and $p \leq z \leq q$ and if f is continuous in the region V, then the integral of f over V is equal to the iterated integral

$$\boxed{\int\limits_W f\, dV = \int\limits_p^q \int\limits_c^d \int\limits_a^b f(x,y,z)\, dx\, dy\, dz}$$

13.1 Triple integrals in Cylindrical coordinates

13.1.1 Cylindrical coordinates

Cylindrical coordinates are defined as the coordinate change

$$\boxed{f : (0, +\infty) \times (-\pi, \pi] \times \mathbb{R} \to \mathbb{R}^3 \setminus \left\{(x, 0, z) \in \mathbb{R}^3 \mid x \leq 0\right\}}$$

given by

$$\boxed{f(r, \varphi, z) = (r\cos\varphi, r\sin\varphi, z)}$$

13.1.2 Integration formula

Each point (x, y, z) in a 3D space is represented by $0 \leq r < \infty,\ -\pi < \varphi \leq \pi$, and $-\infty < z < \infty$. The following relations hold:

$$\boxed{\begin{aligned} &x = r\cos\varphi \\ &y = r\sin\varphi \\ &z = z \\ &x^2 + y^2 = r^2 \end{aligned}}$$

and

$$\boxed{\mathrm{d}V = r\ \mathrm{d}r\ \mathrm{d}\varphi\ \mathrm{d}z}$$

13.2 Changing the Order of Integration

To change the order of integration (e.g., swapping $dy\ dx$ to $dx\ dy$), one must redefine the boundaries of the region Ω. This involves switching from a y-simple description to an x-simple description (or vice versa).

Method:

1. Sketch the region Ω based on the original limits.
2. Identify the boundary curves and rewrite their equations (e.g., convert $y = g(x)$ to $x = g^{-1}(y)$).
3. Determine the new constant limits for the new outer variable.
4. Determine the new variable limits for the new inner variable.

Example: Consider the integral over the region bounded by $y = x^2$, $x = 0$, and $y = 1$:

$$\boxed{\int_0^1 \int_{x^2}^1 f(x, y)\ dy\ dx}$$

To change the order to $dx\ dy$:

- The boundary $y = x^2$ becomes $x = \sqrt{y}$.
- The outer variable y ranges from 0 to 1.
- For a fixed y, x ranges from 0 to $\sqrt{y}$.

$$\boxed{\int_0^1 \int_0^{\sqrt{y}} f(x, y)\ dx\ dy}$$