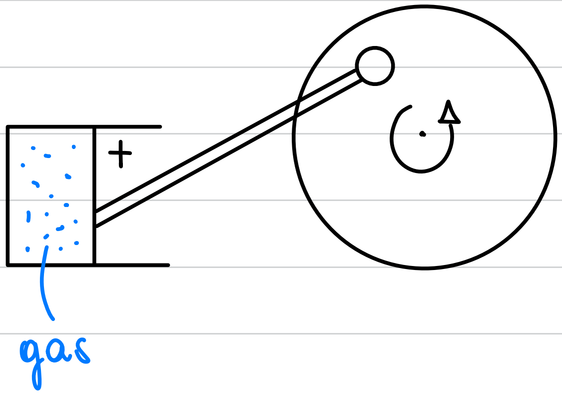
HEAT

Thermal engine



How it works:

1. Working medium, gas
2. Relation $T \leftrightarrow P$: Ideal gas law
3. Heat capacity
4. Mechanical work (W)
5. Efficiency $\eta = \frac{W}{Q}$

Thermodynamic observables - Ideal gas law

P: pressure [Pa]

T: temperature [K]

V: volume [m^3]

n: amount of substance [mol]

R: universal gas constant [J/mol·K]

$= 8{,}314$ J/mol K

$$\boxed{PV = nRT} \rightarrow$$

proportionalities

$p \propto T$	$V \propto T$
$p \propto \frac{1}{V}$	$p \propto N = nR$

Mole

$$N_A = 6{,}02 \cdot 10^{23} \text{ mol}^{-1} \rightarrow \boxed{n = \frac{N}{N_A}}$$

Ideal gasses considerations

- The molecules are point masses, no volume of their own
- There are no attractive forces between the molecules

$\Rightarrow$ Ideal gas law is a good approximation as long as gas is sufficiently dilute

Variation of the ideal gas law

$$\boxed{\begin{gathered} pV = nRT = \frac{N}{N_A} RT = \frac{m}{m_{mol}} RT \\ p = \frac{mRT}{V\, m_{mol}} = \rho \frac{RT}{m_{mol}} \end{gathered}}$$

Roughly air density

Air ≈ 80% N_2, 20% O_2

$$m_{mol,air} = 0{,}8 \cdot m_{mol,N_2} + 0{,}2 \cdot m_{mol,O_2}$$

$$= 0{,}028\ ^{kg}/_{mol} + 0{,}032\ ^{kg}/_{mol} = \boxed{\begin{gathered}0{,}0288\ ^{kg}/_{mol} \\ = 28{,}8\ g/mol\end{gathered}}$$

Heat capacity

$Q = m \cdot c \cdot \Delta T$ [J]

Water: $c_p = 1cal = 4187\ J/kgK$

Oil: $c_p = 1970\ J/kgK$

Heat capacity of gases

$Q = n \cdot c_{(n)} \cdot \Delta T$ where $c_{(n)}$ = moles heat capacity

$$= n \frac{\#DoF}{2} RT$$

Kinetic energy of gases

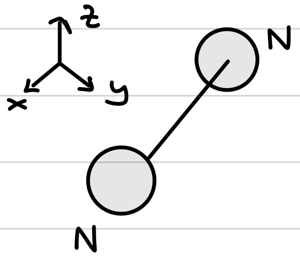
$$E_{KIN} = \sum_{i=1}^{N} \frac{1}{2} m_{(n)} \cdot v_i^2 = \ldots = \boxed{\frac{3}{2} nRT}$$

$$Q = \Delta E_{k,gas} = \boxed{\frac{3}{2} nR\Delta T} \iff Q = n \cdot c_{(n)} \cdot \Delta T = U$$

$$c_{(n)} = \frac{3}{2} R \cong 12{,}47\ J/molK$$

↳ True for all mono-atomic gases (He, Xe, Ne, ...)

Diatomic gases



$$c_{(n)} = \frac{5}{2} R \cong 20{,}785\ J/molK$$

Poliatomic gases

$$\boxed{\begin{gathered} E_{gas} = U = \frac{\#DF}{2} nRT \\ c_{(n)} = \frac{\#DF}{2} R \end{gathered}}$$

where #DF: degrees of freedom

Molecules speed (as point masses) - RMS (Root-Mean-Square)

$$\langle v \rangle = \frac{\sum_n p_n v_n}{\sum_n v_n} \quad ; \quad v_{rms} = \sqrt{\frac{\sum_n p_n^2 v_n}{\sum_n v_n}} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{\sum_n v_n^2}{n}}$$

$$\langle v \rangle = \boxed{\frac{\sum_n v_n}{n}}$$

$$v_{rms} = \sqrt{\frac{3 \underline{k_B} T}{m_{cul}}} = \boxed{\sqrt{\frac{3RT}{m_{mol}}}}$$

Boltzmann constant k_B

$$k_B = \frac{R}{N_A} \cong 1{,}38 \cdot 10^{-23}\ J/K$$

Equipartition principle

At temperature T, the thermal energy $E_{GAS} = n \cdot c_V \cdot \Delta T$ is distributed equally among the DoF in average:

$$\boxed{\frac{\langle E_{part} \rangle}{\#DoF} = \frac{1}{2} k_B T \quad ; \quad \frac{E_{mol}}{\#DoF} = \frac{1}{2} RT}$$

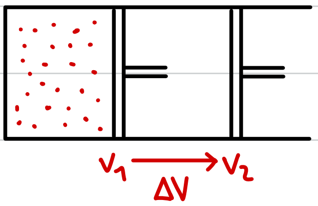
Thermal work

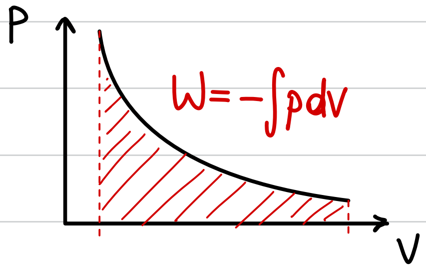
$$W = F \cdot dx \Rightarrow \boxed{W = p \cdot A \cdot dx = -p dV}$$

$$\begin{cases} W < 0, \text{ if } dV > 0 \rightarrow \text{the gas "loses" work} \\ W > 0, \text{ if } dV < 0 \rightarrow \text{the gas "gains" work} \end{cases}$$

Real volume changes → $p \neq$ const

$$\boxed{W = -\int_{V_1}^{V_2} p dV}$$





First law of thermodynamics:
Internal energy

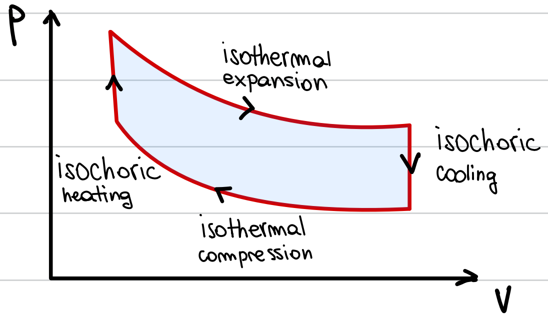
$$\boxed{\begin{aligned} \Delta U_{(U_2 - U_1)} &= Q - W \\ dU = dQ - dW &= dQ - p dV \end{aligned}}$$

$$Q = \Delta U - W = 0$$

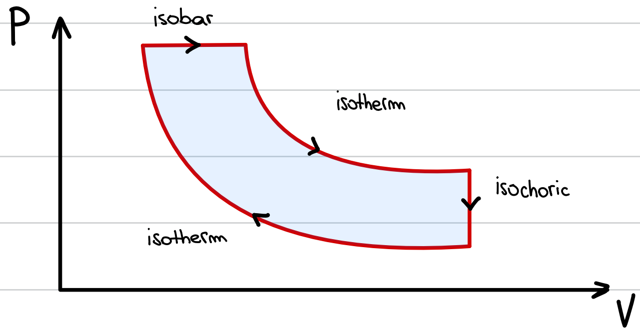
$\Rightarrow \Delta U = W < 0 \rightarrow$ expansion

$\Delta U < 0 \rightarrow$ cooling of the gas

Stirling process



Diesel engine



Blue area: $|W_{net}| = |Q_\alpha - Q_w|$

Specific heat capacity

Molar heat capacity

$$C_p = C_v + R$$

Adiabatic index

$$\gamma = \frac{C_p}{C_v} = \frac{C_v + R}{C_v} = \frac{\#DoF + 2}{\#DoF}$$

$$\Rightarrow \#DoF = \frac{2}{\gamma - 1}$$

Heat capacity at constant pressure

$$\underline{Q_{isochoric}} = \Delta U_{isochoric} - \overset{0}{\cancel{W_{isochoric}}}$$

$$= \boxed{\Delta U_{isochoric} = n c_v \Delta T}$$

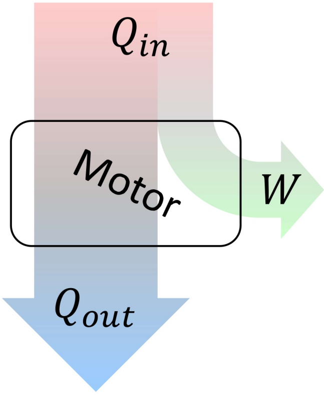
$$\underline{Q_{isobaric}} = \Delta U_{isobaric} - W_{isobaric}$$

$$= n c_v \Delta T - (-p \Delta V) = n c_v \Delta T + p \Delta V$$

$$\boxed{= n \Delta T (c_v + R)}$$

given by the work for expanding the gas

Thermal efficiency



$$\boxed{\eta = \frac{|W_{net}|}{Q_{IN}} = 1 - \frac{|Q_{OUT}|}{Q_{IN}} = \frac{Q_{IN} - |Q_{OUT}|}{Q_{IN}}}$$

$$Q_{IN} = |W_{net}| + |Q_{OUT}| \Rightarrow |W_{net}| = Q_{IN} - |Q_{OUT}|$$

How to calculate thermal efficiency

1. Identify the heat flow

2. Choose a formula for η

Adiabatic process of the ideal gas

$T_1 V_1^{(\gamma-1)} = T_2 V_2^{(\gamma-1)}$ (isobaric)

$P_1 V_1^{\gamma} = P_2 V_2^{\gamma}$ (isothermal)

$T_1 P_1^{\left(\frac{1-\gamma}{\gamma}\right)} = T_2 P_2^{\left(\frac{1-\gamma}{\gamma}\right)}$ (isochoric)

Adiabatic index

$$\gamma = \frac{C_P}{C_V}$$

$$\gamma \begin{cases} \frac{5/2\,R}{3/2\,R} = 1{,}67, & \text{for 1-atomic gas} \\ \frac{7/2\,R}{5/2\,R} = 1{,}4, & \text{for 2-atomic gas} \end{cases}$$

Adiabatic energy

$$\Delta U = nC_V(T_2 - T_1) = \frac{1}{\gamma - 1}(P_2V_2 - P_1V_1) = W_{adiabatic}$$

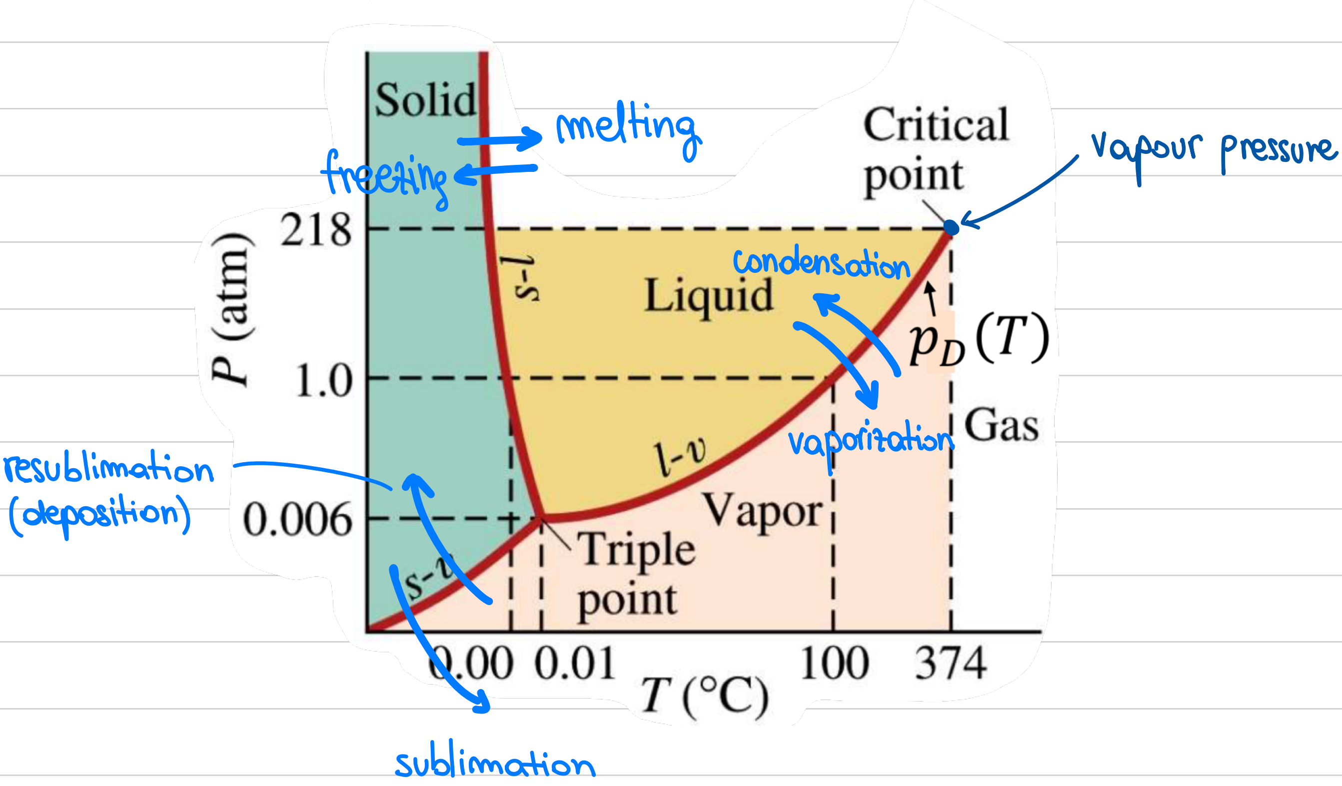
Coefficient of operative performance (COP)

$$COP_{refrigerator} = \frac{Q_{cold}}{W_{net}}$$

$$COP_{heat\ pump} = \frac{Q_{hot}}{W_{net}} = \frac{Q_{cold} + W_{net}}{W_{net}} = 1 + \frac{Q_{cold}}{W_{net}} \Rightarrow COP_{HP} > 1$$

Carnot cycle

Phase diagram (p-T diagram)

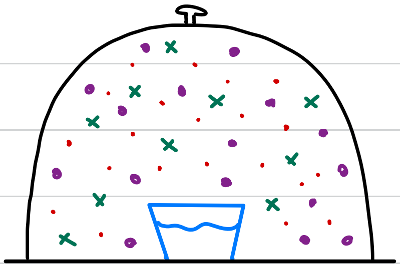


Boiling

Vaporizing

Evaporation

Is vaporization below the boiling point, where water molecules from the liquid phase go into gas phase



× : N_2-molecules → p_{N_2}: partial pressure of N_2

● : O_2-molecules → p_{O_2}: partial pressure of O_2

· : H_2O-molecules → p_{H_2O}: partial pressure of H_2O

$$p_D(t) = \sum p_x \overset{\text{IN OUR CASE}}{=} p_{N_2} + p_{O_2} + p_{H_2O}$$

Equilibrium:

In order for a water surface to be in equilibrium with humid air, the partial pressure p_{H_2O} MUST be equal to the vapour pressure:

$$\boxed{p_{H_2O} = p_D(t)}$$

With this, the saturation of the air has reached 100% RH

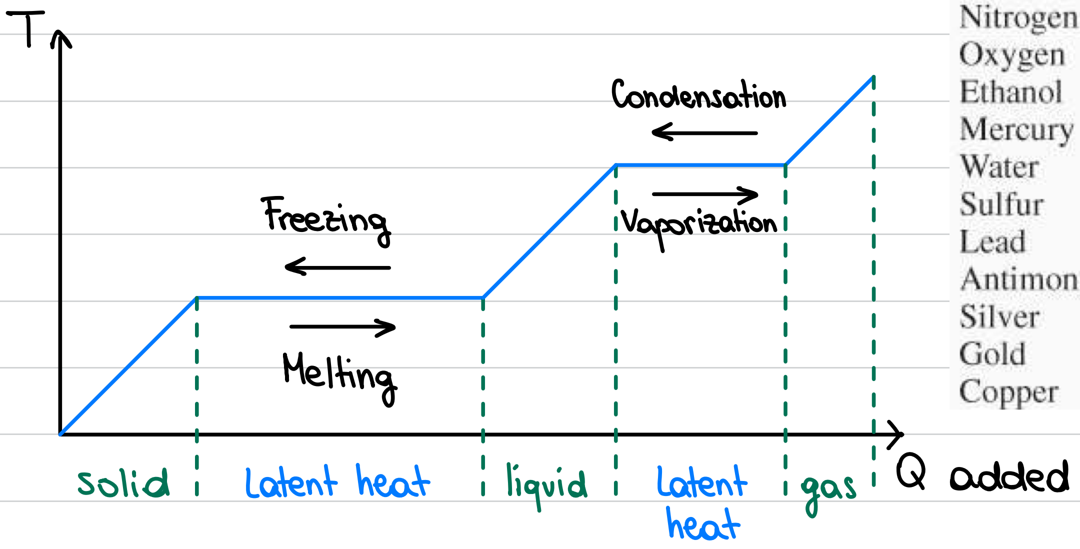
Relativity humidity

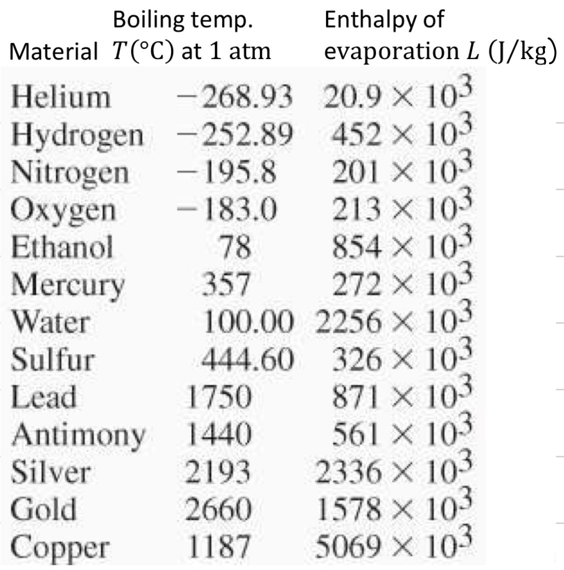
$$RH = \frac{p_{H_2O}}{p_D(t)}$$

Vapour pressure curve of water

T(°C)	p_D(Pa)	T(°C)	p_D(Pa)
0	611	80	47300
5	872	90	70100
10	1227	100	101300
15	1704	110	143200
20	2337	120	198500
25	3168	130	270000
30	4242	150	476000
40	7378	170	791900
50	12300	190	1.255 E6
60	19900	300	8.289 E6
70	31100	350	1.633 E7

Latent heat





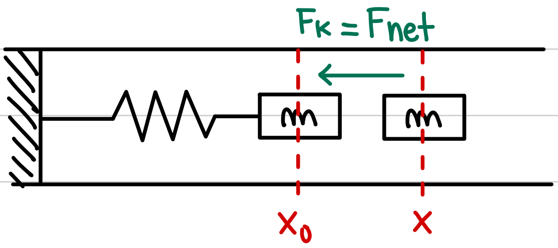
Material	Boiling temp. T(°C) at 1 atm	Enthalpy of evaporation L (J/kg)
Helium	−268.93	20.9×10^3
Hydrogen	−252.89	452×10^3
Nitrogen	−195.8	201×10^3
Oxygen	−183.0	213×10^3
Ethanol	78	854×10^3
Mercury	357	272×10^3
Water	100.00	2256×10^3
Sulfur	444.60	326×10^3
Lead	1750	871×10^3
Antimony	1440	561×10^3
Silver	2193	2336×10^3
Gold	2660	1578×10^3
Copper	1187	5069×10^3

$$Q_{l \to g} = Q_{lat} \cdot m$$

OSCILLATIONS

Motion

Simple harmonic oscillation (SHM)



$$\left.\begin{aligned} F_K &= -k\cdot x \\ F_{net} &= m\cdot\ddot{x} \end{aligned}\right\} \; m\ddot{x} + kx = 0$$

$$\ddot{x} + \frac{k}{m}\cdot x = 0 \qquad \omega = \sqrt{\frac{k}{m}}$$

Dynamics

$$x(t) = A\cdot\cos(\omega t + \varphi)$$

$$v(t) = -\omega A\sin(\omega t+\varphi) = \pm\omega\sqrt{A^2 - x^2}$$

$$a(t) = -\omega^2 A\cos(\omega t+\varphi) = -\omega^2 x(t)$$

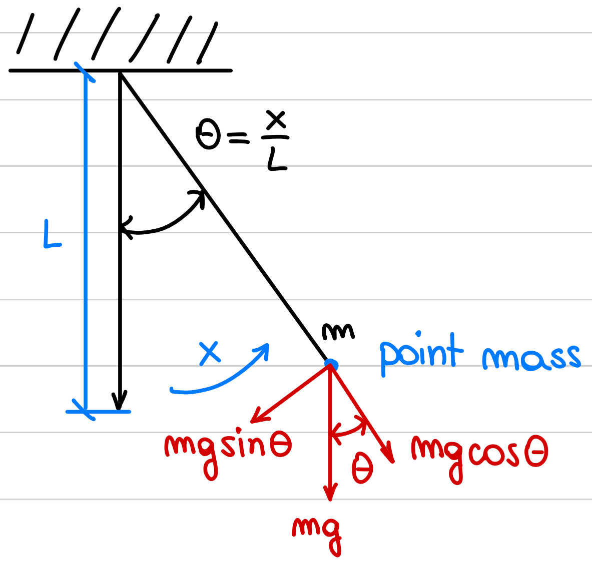
Energy of SHM

$$E_{TOT} = E_{Hooke} + E_{kin} = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

for turning points: $v=0$, $x=A$,

hence: $E_{TOT} = \frac{1}{2}kA^2 + 0 = \boxed{\frac{1}{2}kA^2 = E_{SHM}}$

Simple pendulum



Equation of motion

$$F_{net} = F_{g\parallel} \Leftrightarrow m\ddot{x} = -mg\sin\theta$$

$$\ddot{x} = -g\sin(x/L)$$

$$0 = \ddot{x} + g\sin(x/L)$$

For small θ → $\sin\theta \cong \theta$

Hence: $\ddot{x} + g \cdot x/L = 0$

$$\ddot{x} + x\omega^2 = 0$$

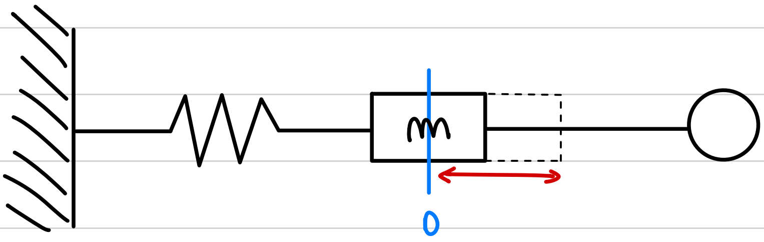
$$\boxed{\omega = \sqrt{g/L} \quad , \quad T = 2\pi\sqrt{L/g}}$$

Phase shift

$\varphi = 0$ if the mass starts oscillating from a rest position

$\varphi \neq 0$ if we have an initial velocity or displacement

Damped oscillations



$$\vec{F}_{net} = \vec{F}_{Hooke} + \vec{F}_{damp} \Leftrightarrow m\ddot{x} = -kx - b\overset{v=\dot{x}}{v}$$

$$\vec{F}_{damp} = \vec{F}_{Stokes} = \underbrace{6\pi R\eta}_{b}v = bv$$

$$\Rightarrow \ddot{x} + \frac{k}{m}x + \frac{b}{m}\dot{x} = 0 \rightarrow \text{No SHM!}$$

Decay constant β

$\beta = \frac{b}{2m}\ [s^{-1}]$

Decay constant

$\tau = \frac{1}{\beta} = \frac{2m}{b}$

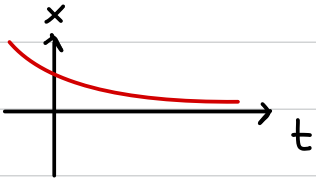
Amplitude

$A(t) = Ae^{-\beta t} = Ae^{-t/\tau}$

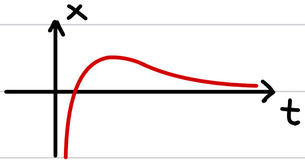
$\Rightarrow \ddot{x} + \omega^2 x + 2\beta\dot{x} = 0$

Overdamped system

$\beta > \omega \Rightarrow x(t) = e^{-\beta t}\left(C_1 e^{\delta t} + C_2 e^{-\delta t}\right)$

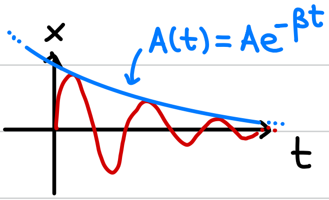


with 1 turning point:



Damped system

$\beta < \omega \Rightarrow x(t) = Ae^{-\beta t}\cos(\omega_{damp}\cdot t + \varphi)$, where $\boxed{\omega_d = \sqrt{\omega^2 - \beta^2},\ \omega_d < \omega}$



Limiting case / Critical damping system

$\beta = \omega \Rightarrow x(t) = e^{-\beta t}(C_1\cdot t + C_2)$

Case of a car shock absorbers

$b_{crit} = 2m\sqrt{\frac{K}{m}} = \sqrt{4km} = 6\pi R\eta$

Energy loss due to damping

$$\langle E(t)\rangle = \frac{1}{2}kA_0^2\, e^{-2t/\tau} = E_0 e^{-2t/\tau}$$

$$= \frac{1}{2}kA_0^2\, e^{-2\beta t} = E_0 e^{-2\beta t}$$

Quality factor

$$Q = \frac{2\pi E(t)}{|\Delta E(t)_{T_D}|}$$

Driven oscillations

Resonance frequency

$$\Omega_R=\sqrt{\omega^2-2\beta^2}\cong\omega\left(1-\frac{1}{4Q^2}\right)$$

Quality factor

$$Q=\pi\cdot\frac{\tau}{\underbrace{T_D}}=\frac{\Omega_R}{\Delta\Omega}=\frac{\Omega_R\sqrt{2}}{A_R}$$

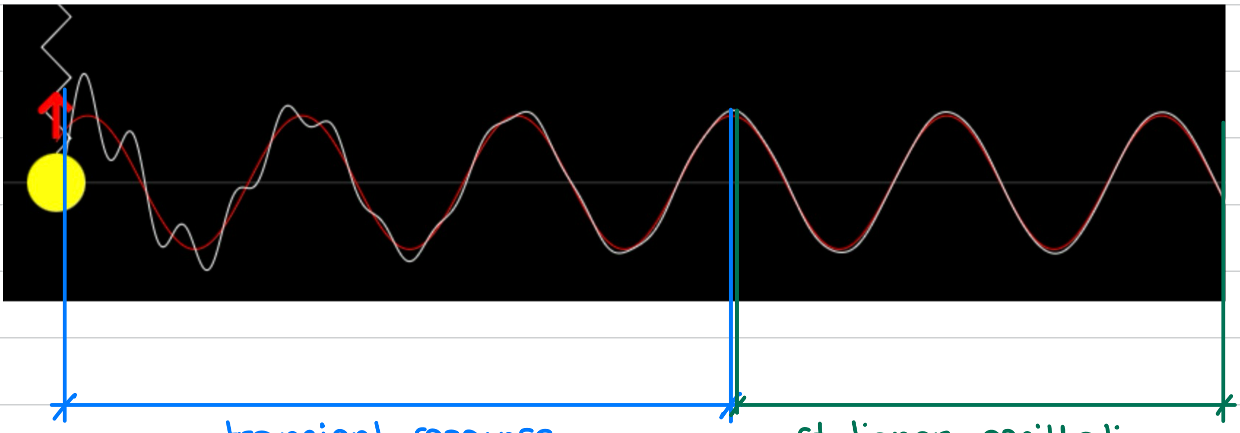
Q is only defined when $\beta<\omega$

Amplitude in resonance (Maximum in resonance curve)

$$A_R=A(\Omega_R)=\frac{K\cdot H}{b\sqrt{\omega^2-\beta^2}}\cong Q\cdot H\cong Q\cdot A_0$$

Position of a driven oscillation

$$y(t)=\underbrace{A_0e^{-\beta t}\cos(\omega_d t+\varphi)}_{\text{transient response}}+\underbrace{A(\Omega)\cos(\Omega t-\phi(\Omega))}_{\text{stationary oscillation}}$$



WAVES

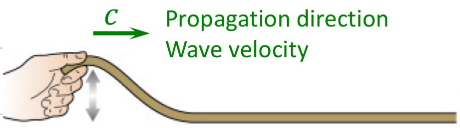
Definition

A wave is an oscillation that propagates through space

Mechanical wave

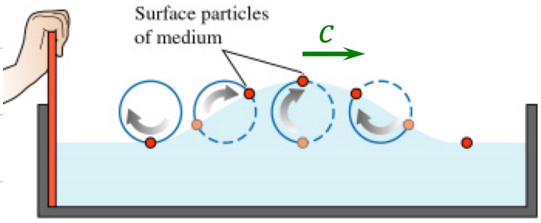
Mass elements oscillating, with mechanical coupling (eg. ropes, soundwaves, ...)

Transverse wave



Longitudinal

Water wave



Wave speed

Rope wave

$$c = \sqrt{\frac{F_s}{\mu}} = \sqrt{\frac{F_s}{\rho \cdot S}}$$

where

F_s: Rope tension [N]

μ: Length density [kg/m]

ρ : Volume density [kg/m^3]

S: cross-sectional area [m^2]

Sound waves

$$c = \sqrt{\frac{\gamma RT}{m_{mol}}}$$

Light waves

$$c = \sqrt{\frac{1}{\varepsilon_0 \mu_0}}$$

where:

ε_0: Dielectric constant

μ_0: Magnetic constant

Harmonic waves

Displacement

$$y(x,t) = A\cos\left(\frac{2\pi}{\lambda}(x-ct)+\varphi\right)$$

$$= Ae^{ik(x-ct)} \quad , \quad i = \sqrt{-1}$$

Properties

$$\omega = \frac{2\pi}{f} \quad , \quad k = \frac{2\pi}{\lambda} \quad ,$$

$$c = \lambda f = \frac{\lambda}{T} = \frac{\omega}{k}$$

Sound wave as a pressure wave

$$p(x,t) = p_{atm} + \hat{p}\cos(kx - \omega t + \varphi) \quad [Pa]$$

Sound pressure amplitude

$$\hat{p} = \rho c \underbrace{\omega A}_{v_{max}} \Leftrightarrow A = \frac{\hat{p}}{\rho \omega c}$$

Energy flux (Power) of an harmonic wave

$$P \propto A^2$$

$$P = \frac{1}{2}\mu\omega^2 A^2 \cdot c = \frac{1}{2}\rho S \omega^2 A^2 c = \frac{S\hat{p}^2}{2\rho c}$$

Intensity of a wave

$$I \propto A^2$$

$$I = \frac{\langle P \rangle}{S} \quad [W/m^2]$$

Sound level

$$L = 10\log\left(\frac{I}{I_0}\right) \quad [dB]$$

where $I_0 = 10^{-12}\left[\frac{W}{m^2}\right]$

Sound, source	Intensity	Sound level
Threshold of pain	25 W/m²	134 dB
Jackhammer, 1 m away / Discotheque	0.01 W/m²	≈ 100 dB
Hearing damage after long term exposure	3.2×10^{-4} W/m²	ab 85 dB
Main road, 10 m away	10^{-4} - 10^{-3} W/m²	80-90 dB
Normal conversation, 1 m away	10^{-8} - 10^{-7} W/m²	40-50 dB
Whisper	10^{-10} W/m²	20 dB
Rustling leaves	10^{-11} W/m²	10 dB

Doppler effect

$$f_{receiver} = \frac{c}{\lambda}, \quad f_{sender} = \frac{c + v_{receiver}}{\lambda}$$

Frequency shift

$$f_r = f_s \cdot \frac{c \pm v_r}{c \mp v_s}$$

$$f_{r\rightarrow\leftarrow s} = f_s \cdot \frac{c + v_r}{c - v_s}, \quad f_{r\leftarrow\rightarrow s} = f_s \cdot \frac{c - v_r}{c + v_s}$$

$$f_{r\rightarrow\rightarrow s} = f_s \cdot \frac{c + v_r}{c + v_s}, \quad f_{r\leftarrow\leftarrow s} = f_s \cdot \frac{c - v_r}{c - v_s}$$

Beat frequency

$$f_B = |f_1 - f_2| = \frac{\Delta\omega}{2\pi}$$

Standing waves (Stationary oscillations)

Hard reflection

$$m_{medium} \lll m_{boundary}$$

$$y_{TOT}(x,t) = y_{IN}(x,t) + y_{OUT}(x,y)$$

$$= \underbrace{-2A\sin(\omega t)}_{\text{common phase}} \cdot \sin(kx)$$

Soft reflection

$$m_{medium} < m_{boundary}$$

$$y_{TOT}(x,t) = y_{IN}(x,t) + y_{OUT}(x,y)$$

$$= \underbrace{2A\sin(\omega t)}_{\text{common phase}} \cdot \cos(kx)$$

Standing wave's nodes

n = number of nodes , $n \in \mathbb{N}$

Node position

$$x_n = n\frac{\lambda_{TOT}}{2}$$

Wavelength

$$\lambda_n = \frac{2L}{n} = \frac{\lambda_1}{n}$$

Frequency

$$f_n = \frac{c}{\lambda_n} = \frac{nc}{2L} = nf_1$$