

I. Navier-Stokes and turbulence modeling

Recap from EFPLab and FLDYN

Conservation laws

$$\dot{m}_\alpha = \dot{m}_\omega$$

$$\sum F = \dot{m} (u_\omega - u_\alpha)$$

Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0$$

Control volumes integration

$$\frac{d}{dt} \int_\Omega \rho dV + \int_\Omega \nabla \cdot \rho \underline{u} dV = 0$$

In steady state:

$$\int_\Omega \nabla \cdot \rho \underline{u} dV = \oint_\Omega \rho \underline{u} \cdot \underline{n} dA = 0$$

For any discrete cell:

$$\oint_{\partial\Omega} \rho \underline{u} \cdot \underline{n} dA = \sum_f \rho_f \underline{u}_f \cdot \underline{n}_f A_f = \sum_f \rho_f \underline{u}_f S_f = 0$$

General form of transportation equation

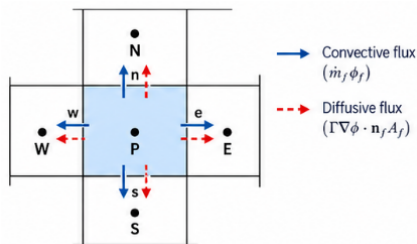
$$\underbrace{\frac{\partial(\rho\Phi)}{\partial t}}_{\text{Accumulation}} + \underbrace{\nabla \cdot (\rho \underline{u} \Phi)}_{\text{Convection}} = \underbrace{\nabla \cdot (\Gamma \nabla \Phi)}_{\text{Diffusion}} + \underbrace{S_\Phi}_{\text{Sources and Sinks}}$$

Finite-volume discretization

$$\underbrace{\frac{d}{dt} (\rho_p \Phi_p V_p)}_{\text{Accumulation}} + \underbrace{\sum_f \dot{m}_f \Phi_f}_{\text{Convection}} = \underbrace{\sum_f (\Gamma_f \nabla \Phi_f \cdot \underline{n}_f A_f)}_{\text{Diffusion}} + \underbrace{S_\Phi V_p}_{\text{Sources and Sinks}}$$

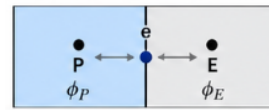
FVD Algorithm

1: Apply conservation to one cell



Apply mass, momentum, energy and turbulence balances to each control volume.

2: Approximate the face fluxes



Approximate face values from cell-centred values: use upwinding for convection and a gradient for diffusion.

$$\nabla \Phi_e = \frac{\partial \Phi}{\partial x} \Big|_e \approx \frac{\Phi_E - \Phi_P}{\Delta x_{PE}}$$

The face flux combines convection and diffusion.

3: Assemble the algebraic equation

$$a_P \phi_P = \sum_{nb} a_{nb} \phi_{nb} + b$$

One equation per cell forms a coupled linear system for pressure, velocity, temperature, etc. Iterate until convergence.

CFD Simulation

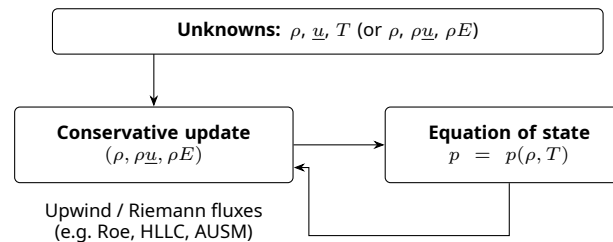
Continuity equation

$$\frac{\partial \rho \underline{u}}{\partial t} + \nabla \cdot \rho \underline{u} \underline{u} = -\nabla p + \nabla \cdot \underline{\tau}$$

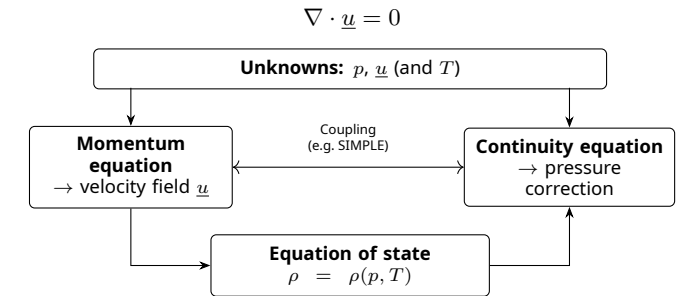
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0$$

Density based

$$p = \rho R T$$



Pressure based



Segregated vs. Coupled solution methods

Segregated method (OpenFOAM, Fluent)

Solve for ∂p :

$$a_c^p p_c + \sum_{nb} a_{nb}^p p_{nb} = b_c^p$$

Solve for ∂u :

$$a_c^{ux} u_{x,c} + \sum_{nb} a_{nb}^{ux} u_{x,nb} = b_c^{ux}$$

$$a_c^{uy} u_{y,c} + \sum_{nb} a_{nb}^{uy} u_{y,nb} = b_c^{uy}$$

$$a_c^{uz} u_{z,c} + \sum_{nb} a_{nb}^{uz} u_{z,nb} = b_c^{uz}$$

- Iterative coupling; often needs underrelaxation and converges slowly.
- Sometimes preferable for transient simulations.
- Fluent: switch between segregated and coupled methods.

Coupled formulation (CFX, Fluent)

- Assembles all four equations in a single coefficient matrix

$$\begin{matrix} \text{Momentum in } x \\ \text{Momentum in } y \\ \text{Momentum in } z \\ \text{Continuity} \end{matrix} \rightarrow \underbrace{\begin{bmatrix} a_c^{uu} & a_c^{uv} & a_c^{uw} & a_c^{up} \\ a_c^{vu} & a_c^{vv} & a_c^{vw} & a_c^{vp} \\ a_c^{wu} & a_c^{wv} & a_c^{ww} & a_c^{wp} \\ a_c^{pu} & a_c^{pv} & a_c^{pw} & a_c^{pp} \end{bmatrix}}_{A_c} \underbrace{\begin{pmatrix} u_c \\ v_c \\ w_c \\ p_c \end{pmatrix}}_{x_c} + \sum_{nb=0; nb \neq c}^N \underbrace{\begin{bmatrix} a_{nb}^{uu} & a_{nb}^{uv} & a_{nb}^{uw} & a_{nb}^{up} \\ a_{nb}^{vu} & a_{nb}^{vv} & a_{nb}^{vw} & a_{nb}^{vp} \\ a_{nb}^{wu} & a_{nb}^{wv} & a_{nb}^{ww} & a_{nb}^{wp} \\ a_{nb}^{pu} & a_{nb}^{pv} & a_{nb}^{pw} & a_{nb}^{pp} \end{bmatrix}}_{A_{nb}} \underbrace{\begin{pmatrix} u_{nb} \\ v_{nb} \\ w_{nb} \\ p_{nb} \end{pmatrix}}_{x_{nb}} = \underbrace{\begin{pmatrix} b_c^u \\ b_c^v \\ b_c^w \\ b_c^p \end{pmatrix}}_{b_c}$$

- Can become memory expensive
- Requires more care in implementation

Turbulence modeling**Turbulent flow development****Reynolds number**

$$Re = \frac{\rho UL}{\mu} = \frac{UL}{\nu}$$

- Compares inertial and viscous effects.
- Mainly used for forced-convection flows.

Grashof number

$$Gr = \frac{g\beta\Delta TL^3}{\nu^2}$$

- Compares buoyancy and viscous effects.
- Important for natural-convection flows.

Rayleigh number

$$Ra = Gr Pr$$

- Often used for transition in buoyancy-driven heat transfer.

Typical transition criteria

Flow type	Criterion	Typical onset
Pipe flow	Re_D	Laminar below 2300; transition ~ 2300 –4000; turbulence above 4000.
Flat-plate boundary layer	Re_x	$\sim 5 \times 10^5$
Cylinder wake	Re_D	Vortex shedding around 47; transition to a turbulent wake much later.
Natural convection along a vertical plate	Ra_x	Often around $Ra_x > 10^9$.

Inertial subrange

Between large eddies and dissipative eddies, the spectrum depends only on the transfer rate ϵ and the wavenumber k .

$$E(k) = C_K \epsilon^{2/3} k^{-5/3}, \quad C_K \approx 1.5$$

Kolmogorov scale

The far right of the spectrum, where viscosity finally acts.

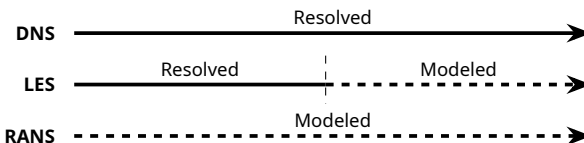
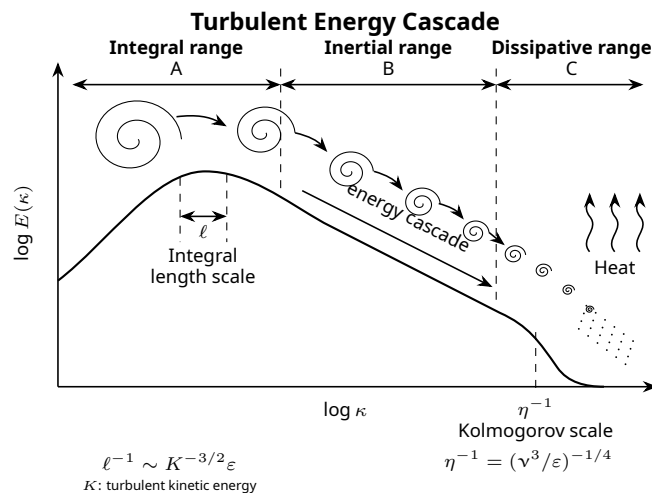
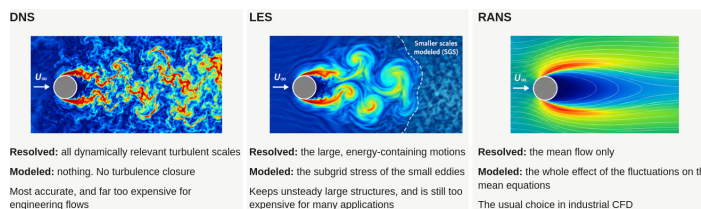
$$\eta = \left(\frac{\nu^3}{\epsilon} \right)^{1/4}$$

where: η smallest eddy size [m]
 ν kinematic viscosity [m²/s]
 ϵ dissipation rate of kin. energy [m²/s³]

Turbulent kinetic energy and the three ranges

$$k = \int_0^\infty E(k) dk$$

The area under the spectrum is k

**DNS, LES, and RANS****Reynolds-averaged Navier Stokes (RANS)**

Ensemble averaging extracts the mean from instantaneous flow properties:

$$\underbrace{u_i(x, t)}_{\text{instantaneous}} = \underbrace{\overline{u_i(x, t)}}_{\text{average}} + \underbrace{u'_i(x, t)}_{\text{fluctuation}}$$

For incompressible flow with constant ρ and μ , pressure and viscous stress are linear and average without a new unknown. Only the convective term is nonlinear.

1. Instantaneous momentum

$$\rho \left(\frac{\partial u_i}{\partial t} + \underbrace{u_j \frac{\partial u_i}{\partial x_j}}_{\text{nonlinear}} \right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

2. Decompose the product and average

$$u_i = \overline{u_i} + u'_i,$$

$$u_i u_j = \overline{u_i} \overline{u_j} + \overline{u_i u'_j} + \overline{u'_i u_j} + \overline{u'_i u'_j},$$

$$\overline{u_i u_j} = \overline{u_i} \overline{u_j} + \overline{u'_i u'_j}$$

The cross terms vanish because $\overline{u'_i} = 0$.

3. Averaged momentum

$$\rho \left(\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j} \frac{\partial \overline{u_i}}{\partial x_j} \right) = -\frac{\partial \overline{p}}{\partial x_i} + \mu \frac{\partial^2 \overline{u_i}}{\partial x_j \partial x_j} - \underbrace{\frac{\partial}{\partial x_j} (\overline{\rho u'_i u'_j})}_{\text{Reynolds stress}}$$

$$\tau_{ij}^R = -\overline{\rho u'_i u'_j}$$

In 3D, this symmetric tensor has six independent components, but there are only three momentum equations.

Eddy-viscosity models (EVM) provide closure, e.g. k - ϵ , k - ω and SST.

Modeling strategies**Eddy-viscosity models (EVM)**

The six unknown stresses are related to the mean strain, as for a viscous stress.

Boussinesq hypothesis

$$\tau_{ij}^R = -\overline{\rho u'_i u'_j} = 2\mu_t S_{ij} - \frac{2}{3}\rho k \delta_{ij}$$

- $2\mu_t S_{ij}$ aligns the deviatoric stress with the mean strain
- $\frac{2}{3}\rho k \delta_{ij}$ is the isotropic normal-stress contribution
- δ_{ij} is the identity tensor

$$S_{ij} = \frac{1}{2} \left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i} \right), \quad k = \frac{1}{2} \overline{u'_i u'_i}$$

S_{ij} is the mean strain rate and k the turbulent kinetic energy per unit mass.

The eddy viscosity μ_t is modeled, not a fluid property:

- One equation: Spalart-Allmaras.
- Two equations: k - ϵ , k - ω , SST.

Eddy viscosity ratio

$$EVR = \beta = \frac{\eta_t}{\eta}$$

Two-equation models

- Common turbulence models, widely used in industry.
- Two additional transport equations are solved iteratively.
- Usually solve for turbulent kinetic energy k and either dissipation rate ϵ or specific dissipation rate ω .

Turbulent kinetic energy

$$k \equiv \frac{\overline{u'_i u'_i}}{2}$$

Kinetic energy per unit mass of turbulent fluctuations, mainly contained in the large eddies.

Dissipation rate of turbulent kinetic energy

$$\epsilon \equiv \nu \frac{\partial u'_i}{\partial x_j} \left(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i} \right)$$

Rate at which turbulent kinetic energy is converted into heat per unit mass, predominantly at the smallest eddies.

Specific dissipation rate, or turbulent frequency

$$\omega \equiv \frac{\epsilon}{C_\mu k}, \quad C_\mu = 0.09$$

Dissipation rate per unit turbulent kinetic energy, scaled by $1/C_\mu$; an inverse turbulent time scale with units s^{-1} .

Reynolds-averaged momentum equations under EVM**1. Averaged momentum, Reynolds stress still unknown**

$$\frac{\partial \rho u_i}{\partial t} + \nabla \cdot (\rho u_i u_j) = -\nabla p + \nabla \cdot \left[\mu \left(\nabla u_i + \nabla u_i^T \right) \right] - \nabla \cdot \left(\rho \overline{u'_i u'_j} \right)$$

2. Insert the Boussinesq relation

$$-\rho \overline{u'_i u'_j} = 2\mu_t S_{ij} - \frac{2}{3} \rho k I_{ij} = \mu_t \left(\nabla u_i + \nabla u_i^T \right) - \frac{2}{3} \rho k I_{ij}$$

Because $2S_{ij} = \nabla u_i + \nabla u_i^T$, the strain term has the same form as the viscous stress, so μ_t adds to μ .

3. Momentum equation under EVM

$$\frac{\partial \rho u_i}{\partial t} + \nabla \cdot (\rho u_i u_j) = -\nabla p + \nabla \cdot \left[(\mu + \mu_t) \left(\nabla u_i + \nabla u_i^T \right) \right] - \nabla \cdot \left(\frac{2}{3} \rho k I_{ij} \right)$$

Effective viscosity is the sum of physical and modeled viscosity:

$$\mu_{\text{eff}} = \mu + \mu_t$$

EVM-Model 1: $k - \epsilon$ turbulence model

Has been the most popular turbulence model in the past. Still the reference in many industries due to its robustness.

Turbulent viscosity

$$\mu_t = \rho c_\mu \frac{k^2}{\epsilon}$$

PDE for turbulent kinetic energy

$$\underbrace{\rho U_i \frac{\partial k}{\partial x_i}}_{\text{Convection}} = \underbrace{\mu_t \left(\frac{\partial U_j}{\partial x_i} + \frac{\partial U_i}{\partial x_j} \right) \frac{\partial U_j}{\partial x_i}}_{\text{Production}} + \underbrace{\frac{\partial}{\partial x_i} \left\{ (\mu_t / \sigma_k) \frac{\partial k}{\partial x_i} \right\}}_{\text{Diffusion}} - \underbrace{\frac{\rho \epsilon}{\sigma_\epsilon}}_{\text{Dissipation}}$$

PDE for turbulent dissipation rate

$$\underbrace{\rho U_i \frac{\partial \epsilon}{\partial x_i}}_{\text{Convection}} = \underbrace{C_{1\epsilon} \left(\frac{\epsilon}{k} \right) \mu_t \left(\frac{\partial U_j}{\partial x_i} + \frac{\partial U_i}{\partial x_j} \right) \frac{\partial U_j}{\partial x_i}}_{\text{Production}} + \underbrace{\frac{\partial}{\partial x_i} \left\{ (\mu_t / \sigma_\epsilon) \frac{\partial \epsilon}{\partial x_i} \right\}}_{\text{Diffusion}} - \underbrace{C_{2\epsilon} \rho \left(\frac{\epsilon^2}{k} \right)}_{\text{Dissipation}}$$

$\sigma_k, \sigma_\epsilon, C_{1\epsilon}, C_{2\epsilon}, C_\mu$ are empirical constant, tuned to canonical flows (very simple, well understood cases)

- Good compromise between numerical effort and computational accuracy
- Two transport equations for the solution of TKE and turbulent dissipation rate
- Turbulent viscosity is modeled as product of turbulent velocity and turbulent length scale
- Good predictions for many flows of engineering interest

Not suitable for:

- Flow with boundary layer separation
- Flow with sudden changes in the mean strain rate
- Flows in rotating fluids
- Flows over curved surfaces

EVM-Model 2: $k - \omega$ turbulence model

Developed to address weakness of previous model in near wall treatment for wall bounded flows, no more need for near wall region modifications.

Turbulent viscosity

$$\mu_t = \rho \frac{k}{\omega}$$

ω is an inverse time scale associated with turbulence (mathematically ϵ/k replaced by ω).

PDE for turbulent kinetic energy

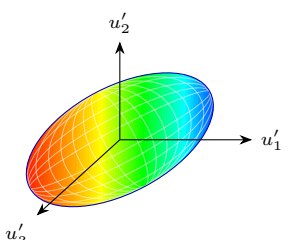
$$\underbrace{\rho U_i \frac{\partial k}{\partial x_i}}_{\text{Convection}} = \underbrace{\mu_t \left(\frac{\partial U_j}{\partial x_i} + \frac{\partial U_i}{\partial x_j} \right) \frac{\partial U_j}{\partial x_i}}_{\text{Production}} + \underbrace{\frac{\partial}{\partial x_i} \left\{ (\mu + \sigma_k \mu_t) \frac{\partial k}{\partial x_i} \right\}}_{\text{Diffusion}} - \underbrace{\beta^* \rho \omega k}_{\text{Dissipation}}$$

PDE for turbulent dissipation rate

$$\underbrace{\rho U_i \frac{\partial \omega}{\partial x_i}}_{\text{Convection}} = \underbrace{\frac{\alpha \omega}{k} \mu_t \left(\frac{\partial U_j}{\partial x_i} + \frac{\partial U_i}{\partial x_j} \right) \frac{\partial U_j}{\partial x_i}}_{\text{Production}} + \underbrace{\frac{\partial}{\partial x_i} \left\{ (\mu + \sigma_\omega \mu_t) \frac{\partial \omega}{\partial x_i} \right\}}_{\text{Diffusion}} - \underbrace{\beta \rho \omega^2}_{\text{Dissipation}}$$

$\beta^*, \beta, \alpha, \sigma_k, \sigma_\omega$ are empirical constant, tuned to canonical flows (very simple, well understood cases)

Reynolds-stress models (RSM)

$$-\rho \overline{u'_i u'_j} = \begin{bmatrix} \tau_{11}^R & \tau_{12}^R & \tau_{13}^R \\ \tau_{21}^R & \tau_{22}^R & \tau_{23}^R \\ \tau_{31}^R & \tau_{32}^R & \tau_{33}^R \end{bmatrix}$$


The stress is not forced to follow S_{ij} .

Each independent component has its own transport equation.

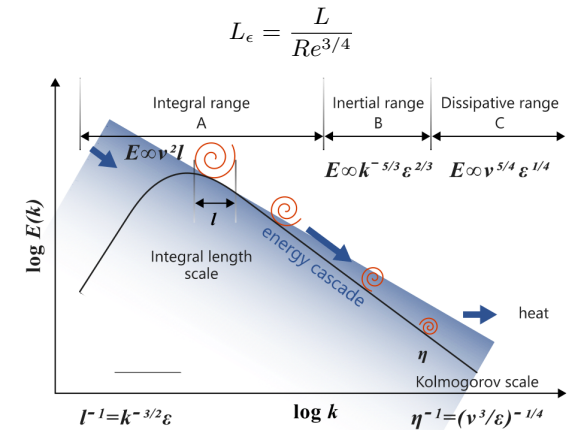
- Symmetric $\underline{\tau}$: six unknowns, six stress transport equations.
- Can represent turbulence anisotropy.
- More computationally demanding.

Both approaches close the RANS equations:

EVM models μ_t ; RSM transports the six stresses.

Energy cascade

- Injection of energy at the largest eddies
- Flux of energy from large-scale eddies toward dissipating eddies
- Dissipation of energy by viscous dissipation
- Generation of largest eddies, then energy transfer, then viscous dissipation



Navier-Stokes equations (NS)

Conservation equations

Name	EFPLab	NS PDE
Continuity	$\dot{m}_\alpha = \dot{m}_\omega$	$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \underline{u} = 0$
Momentum	$\dot{m} \underline{u} = \sum F$ $\dot{m} \underline{u} = \rho \underline{u} \underline{u} A$	$\frac{\partial \rho \underline{u}}{\partial t} + \nabla \cdot \rho \underline{u} \underline{u} = -\nabla p + \nabla \cdot \underline{\underline{\tau}}$ $\underline{\underline{\tau}} = \nabla \cdot \eta \nabla \underline{u}$

$$\underline{\underline{\tau}} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

Steady momentum equation

$$\nabla \rho \underline{u} \underline{u} = -\nabla p + \nabla \cdot \eta \left(\nabla \underline{u} + \nabla \bar{u}^T \right)$$