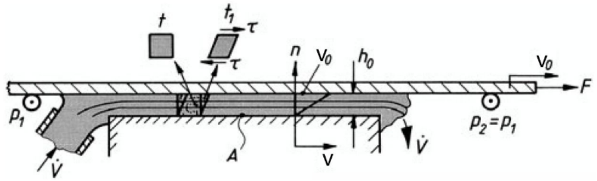
I. Viscous flow

Symbology

Symb	Name	Units	Symb	Name	Units
ν	Kinematic viscosity	m^2/s	η	Dynamic viscosity	Pa·s
τ	Shear stress	Pa	dn	Normal distance	m
du/dy	Rate of strain	[-]	e_{Diss}	Spec. diss. power	W/m^3

Flows with friction



$$F = \eta \frac{v_0 \cdot A}{h_0} \Longrightarrow \tau = \frac{F}{A} = \eta \frac{v_0}{h_0} = \eta \frac{dv}{dn}$$

For pipes:

$$F_\tau = \tau A \Longrightarrow M_\tau = F_\tau y = \tau \cdot l \cdot 2\pi r_0^2$$

Friction law

$$\tau = -\eta \frac{dv}{dn} \quad ; \quad \dot{e}_{\text{Diss}} = \eta \left(\frac{dv}{dn}\right)^2$$

Viscosity

If $\eta = 0$, the fluid is called inviscid.

No slip condition: velocity of a particle closest to the wall has velocity equal to the wall velocity.

Reynolds Number

$$Re = \frac{\rho v L}{\eta} = \frac{vL}{\nu} \quad ; \quad \nu = \frac{\eta}{\rho}$$

Newtonian fluids

- Rate of deformation du/dy (velocity gradient) is linearly proportional to the shear stress τ
- The constant of proportionality is the viscosity (slope)
- $\eta = \dfrac{\tau}{du/dy} = \text{constant}$

Non-Newtonian fluids

Non-linear relation between shear stress and deformation rate due to non-constant viscosity.

$$\tau = \eta\left(\frac{\partial u}{\partial y}\right) \cdot \frac{\partial u}{\partial y}$$

Dilatant (shear thickening) fluids

Viscosity increases with increasing strain rate.

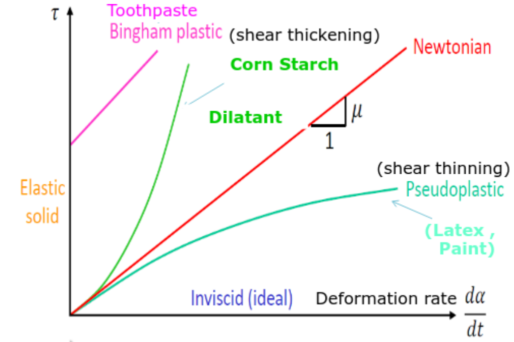
Plastic or pseudo-plastic (shear thinning) fluids

Viscosity decreases with increasing strain rate.

Bingham medium

Flow occurs only after a yield stress τ_0 is reached.

Graphical representation



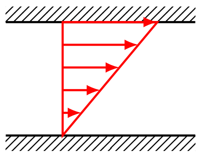
Flow profiles

Couette flow profiles (constant velocity gradient)

$$\frac{\partial u(y)}{\partial y} = \frac{v_p}{s}$$

$$e_{\text{Diss}} = \eta \left(\frac{\partial u(y)}{\partial y}\right)^2 = \eta \left(\frac{v_p}{s}\right)^2$$

$$P_{\text{Diss}} = \int_V e_{\text{Diss}}(y)\, dV$$



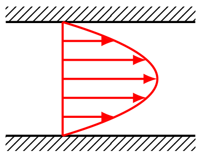
Poiseuille flow profiles (parabolic velocity gradient)

$$\frac{\partial u(y)}{\partial y} = \frac{4 v_{\text{max}}}{s}\left(1 - \frac{2y}{s}\right)$$

$$v(r) = ar^2 + br + c$$

$$e_{\text{Diss}} = \eta \frac{16 v_{\text{max}}^2}{s^2}\left(1 - \frac{2y}{s}\right)^2$$

$$P_{\text{Diss}} = \frac{16 \eta A v_{\text{max}}^2}{3s}$$



Laminar pipe flow

$p_1 > p_2$

$$\frac{\partial p}{\partial z} = \frac{p_2 - p_1}{L}$$

Constant linear flow profiles

$$\dot{V} = v_m A = v_m R^2 \pi$$

Parabolic flow profiles

$$\dot{V} = \int v(r)\, dA = \frac{v_{max} \cdot R^2 \pi}{2}$$

$$v(r) = v_{max}\left(1 - \frac{r^2}{R^2}\right)$$

$$\frac{dv}{dr} = -2 v_{max} \frac{r}{R^2}$$

$$P_{\text{Diss}} = \int_0^L \int_0^{2\pi} \int_0^R \eta \left(-2 v_{max} \frac{r^2}{R^2}\right) \cdot r\, dr\, d\varphi\, dx$$

$$P_{\text{Diss}} = 2\pi L \int_0^R \eta \left(-2 v_{max} \frac{r^2}{R^2}\right) \cdot r\, dr = 2\pi L \eta v_{max}^2 = 8\pi L \eta v_m^2$$

Flows in gaps and bearings

Assumptions in gaps and bearings

- Newtonian fluid with constant viscosity η
- Incompressible fluid
- Very low Reynolds number $Re < 2000$
- Laminar flow
- Balance of pressure and viscous forces
- No slip condition at the walls
- No acceleration (negligible)
- Pressure is uniform across the thickness $\partial p / \partial y \approx 0$

Equilibrium of forces

$$\sum F_x = 0 \Longrightarrow p A_p - (p + dp) A_p + \tau A_\tau + (\tau + d\tau) A_\tau = 0$$

$$A_p = dy \cdot b \quad ; \quad A_\tau = dx \cdot b \quad ; \quad \tau = \eta \frac{dv}{dy} \rightarrow \frac{d\tau}{dy}$$

$$\boxed{\eta \frac{d^2 v(x,y)}{dy^2} = \frac{dp(x)}{dx} = p'}$$

Basic velocity equation

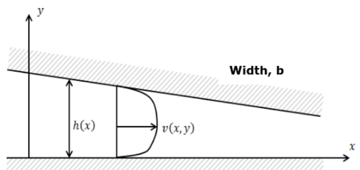
$$v(x,y) = \frac{1}{\eta} \cdot p' \cdot \frac{y^2}{2} + c(x) y + d(x)$$

Gap flow (walls not moving)

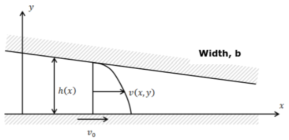
$$v(x,y) = \frac{p'(x)}{2\eta}\left[y^2 - h(x) y\right]$$

$$c(x) = -\frac{h(x)}{2\eta} \cdot \frac{dp(x)}{dx}$$

$$d(x) = 0$$



Plain bearing / Creeping gap flow (one wall moving)



Assumption for $Re \ll 1$: $\eta \frac{d^2 v}{dy^2} = \frac{dp}{dx} = p'$

$$v(x,y) = \frac{p'(x)}{2\eta}\left[y^2 - h(x)y\right] + \frac{v_0}{h(x)}\left[h(x) - y\right]$$

$$c(x) = -\frac{h(x)}{2\eta}\cdot\frac{dp(x)}{dx} - \frac{v_0}{h(x)}$$

$$d(x) = v_0$$

For the simplest, parabolic velocity profile case:

$$h(x) = h_0 \quad ; \quad v_0 = 0 \quad ; \quad p' = \frac{dp(x)}{dx} = \frac{\Delta p}{\Delta L} = -\frac{12\eta v_m}{h_0^2}$$

$$v(y) = \frac{p'}{2\eta}\left(y^2 - h_0 y\right) \Longrightarrow \frac{\partial v}{\partial y} = 0 = \frac{p'}{2\eta}(2y - h_0)$$

$$y = \frac{h_0}{2} \Longrightarrow v_{max} = v\left(y = \frac{h_0}{2}\right) = -\frac{h_0^2 p'}{8\eta} = \frac{3}{2}v_m$$

$$v_m = -\frac{\Delta p \cdot h_0^2}{12\eta L}$$

$$\dot V = b\int_0^{h_0} v(y)\,dy = -b\frac{p' h_0^3}{12\eta} = \frac{h_0^2}{6\eta}b\Delta p$$

General sliding bearings:

$$\dot V = b\int_0^{h(x)} \frac{p'}{2}\eta\left(y^2 - h(x)y\right) + \frac{v_0}{h(x)}\left(h(x) - y\right)\,dy$$

$$\dot V = -b\frac{p' h_{(x)}^3}{12\eta} + b\frac{v_0 h(x)}{2} = ¢$$

$$p' = 6\eta v_0\left(\frac{1}{h(x)^2} - \frac{\dot V}{b h(x)^3}\right)$$

General form of the volumetric flow rate

$$v(x,y) = \frac{1}{\eta}p'\frac{y^2}{2} + c(x) + d(x)$$

$$\dot V = b\int_0^{h(x)} v(x,y)\,dy = b\left[-\frac{h^3(x)}{6\eta}p' + c(x)\frac{h^2(x)}{2} + d(x)h(x)\right] = ¢$$

$$p' = \frac{6\eta}{h(x)^3}\left(\frac{\dot V}{b} - c(x)\frac{h(x)^2}{2} - d(x)h(x)\right)$$

Possible semplifications

If h $\ll$ D:

$$A = \int_A r\,dr\,d\varphi \approx \pi D h$$

Specific dissipation power

$$P = \tau A v = \eta\frac{\partial v}{\partial n}Av \quad ; \quad dP = dF dv$$

$$dF_\tau = d\tau dA = \eta\frac{dv}{dy}dxdz$$

$$e_{\text{Diss}} = \frac{dP_\tau}{dV} = \frac{\eta\frac{dv}{dy}dxdzdv}{dxdydz} = \eta\left(\frac{dv}{dy}\right)^2 = \eta\left(\frac{\partial v}{\partial y}\right)^2$$

$$P_{\text{Diss}} = \int_V e_{\text{Diss}}\,dV = \iiint_V e_{\text{Diss}}\,dxdydz = M\cdot\omega$$

Pressure loss according to Bernoulli

$$p_1 - p_2 = \rho\frac{P_{\text{Diss}}}{\dot m} = \frac{P_{\text{Diss}}}{\dot V} = \frac{8\pi L\eta v_m}{R^2}\ (= \rho g h)$$

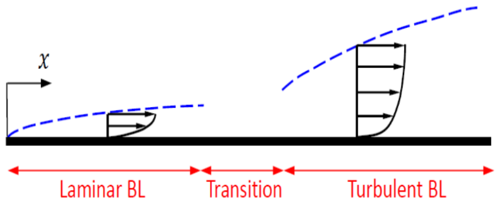
II. Boundary layers

Symb	Name	Units	Symb	Name	Units
v_∞	free-stream velocity	m/s	$v(y)$	local velocity in boundary layer	m/s
δ	boundary layer thickness	m	δ^*	BL displacement thickness	m
θ	momentum thickness	m	τ_w	wall shear stress	Pa
ν	kinematic viscosity	m^2/s	η	dynamic viscosity	Pa·s
Re_x	Reynolds num. at position x	-	$\dot V_{BL}$	boundary-layer volume flow rate	m^2/s

Boundary layers develop due to wall shear stress τ and fluid viscosity ν.

Laminar vs. Turbulent boundary layers

Critical Reynolds number

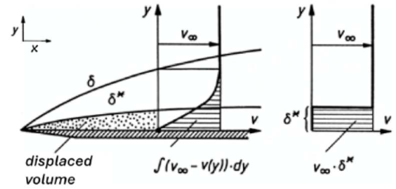


Reynolds number for the BL on a flat plate:

$$Re_x = \frac{v_\infty x}{\nu} = \frac{v_\infty \cdot l_{char}}{\nu}$$

Re **increases** with the distance from the plate entry.
Transition to turbulent: $Re_x \geq 5\times 10^5$

Flat plate boundary layer



$$\delta(x) \approx \sqrt{x} \quad ; \quad \tau(x) \approx 1/\delta(x) = 1/\sqrt{x}$$

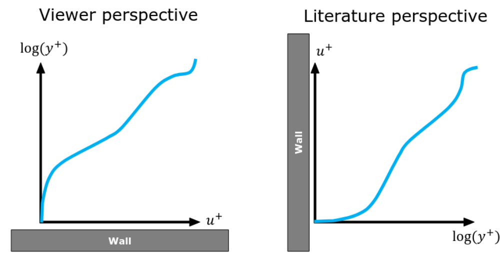
$$\dot V_{BL} = v_\infty \cdot \delta^*(x)$$

with 2 plates: $\dot V_{BL} = A^*\cdot v = \left(\frac{d - 2\delta^*}{2}\right)\pi\cdot v$

Turbolent BL

Non-dimensional forms

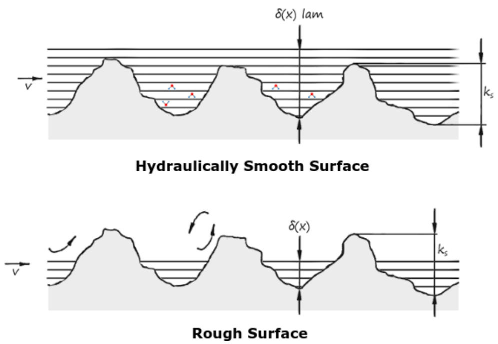
Non-dimensional plot of wall normal turbulent velocity profile:



- Friction velocity: $u_\tau = \sqrt{\frac{\tau_{\text{wall}}}{\rho}}$
- Dimensionless wall parallel velocity: $u^+ = \frac{u}{u_\tau}$
- Dimensionless wall distance: $y^+ = \frac{y u_\tau}{\nu}$

Rough and smooth walls and the effect on friction

No-slip condition



- Hydraulically smooth surface: $\delta(x) > k_s$
- Rough surface: $\delta(x) < k_s$. Leads to vortex.

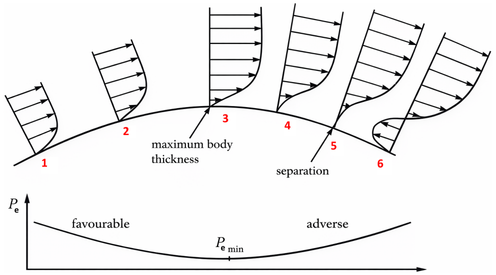
Flow separation

Flow separation conditions

1. Friction
2. Pressure increase

At sharp edges, the flow always separate.

Pressure gradient



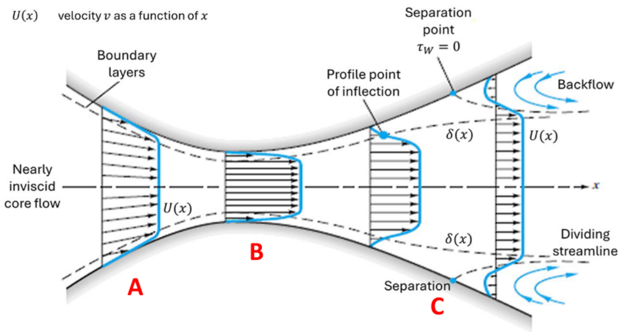
Comparison of boundary layer profile shape

Comparing BL as a function of the pressure gradient:

$$\frac{dP}{dx} \sim -\frac{dU(x)}{dx} \quad ; \quad \tau_{\text{wall}} = m\frac{dv}{dy_{y=0}}$$

1. favorable pressure gradient: $\frac{\partial p}{\partial x} \ll 0$
2. favorable pressure gradient: $\frac{\partial p}{\partial x} < 0$
3. zero pressure gradient: $\frac{\partial p}{\partial x} = 0$
4. mild adverse pressure gradient: $\frac{\partial p}{\partial x} > 0$
5. critical adverse pressure gradient: $\frac{\partial p}{\partial x} > 0, \tau_{\text{wall}} = 0$
6. large adverse pressure gradient: $\frac{\partial p}{\partial x} \gg 0$

Flow separation gradients



Nozzle (Favorable Pressure Gradient) (A)

$$\frac{dp}{dx} < 0 \quad ; \quad \frac{dU}{dx} > 0$$

- Decreasing pressure and decreasing area
- Pressure decreases in the flow direction
- Pressure force is in the flow direction
- No flow separation

Throat (B)

$$\frac{dp}{dx} = 0 \quad ; \quad \frac{dU}{dx} = 0$$

- Constant pressure and constant area
- Fluid particles in the BL slow down due to shear stress only
- No flow separation can occur
- Similar to a flat plate

Diffuser (Adverse Pressure Gradient) (C)

$$\frac{dp}{dx} > 0 \quad ; \quad \frac{dU}{dx} < 0$$

- Pressure increases in the flow direction
- Pressure force is in opposite direction of the flow
- **Back flow situation**: Fluid particles close to the wall with low momentum may come to a stop or even move in opposite direction of the main flow

Influence of Re on blunt bodies

- $Re < 1$: No vortex formation
- $Re \approx 10^5$: Laminar BL, vortex formation in dead water area
- $Re > 10^6$: BL partially turbolent, vortex on the body surface

III. Drag and Lift Forces

Drag force: parallel and opposite to the relative flow direction
Lift force: perpendicular to the relative flow direction

Drag force and drag coefficient

Drag force

$$F_D = C_D q_\infty A_P = C_D \frac{1}{2}\rho v_\infty^2 A_p$$

where:

$$C_D : \text{drag coefficient } (\mu) \quad ; \quad A_p : \text{project area}$$
$$q_\infty : \text{dynamic contribution}$$

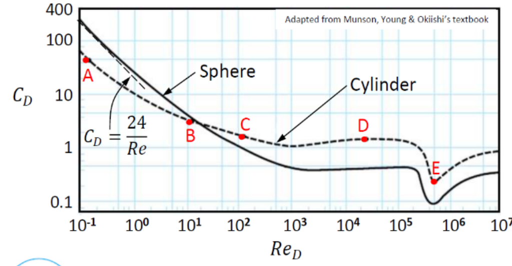
Drag force and power on a body in a flow

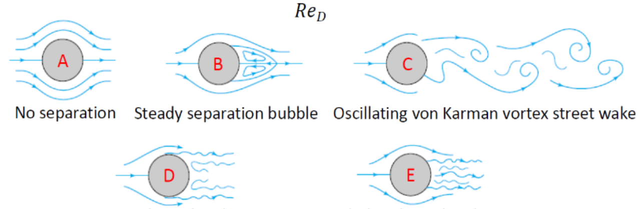
$$F_D = F_P + F_\tau \quad ; \quad P_D = F_D v_\infty$$

Friction drag: friction forces F_τ due to wall shear stress τ_{wall}
Pressure drag: pressure forces F_P due to pressure variation along the surface of the object

Drag on smooth circular cylinders

Prandlt's experiment





No separation — Steady separation bubble — Oscillating von Karman vortex street wake

Laminar boundary layer, wide turbulent wake — Turbulent boundary layer, narrow turbulent wake

Stokes formula and Stokes law

$$F_D \approx F_\tau = 3\pi\eta d v_\infty \implies C_D = \frac{24}{Re}$$

Velocity of a sphere

$$\frac{1}{2}C_D \rho_{\text{air}} A v_\infty = \rho_{\text{sph}} V_{\text{sph}} g \quad ; \quad A = \left(\frac{d}{2}\right)^2 \pi \quad ; \quad V = \left(\frac{d}{2}\right)^3 \pi$$

$$v_\infty = \sqrt{\frac{4}{3}\frac{d \cdot g \cdot \rho_{\text{sph}}}{C_D \cdot \rho_{\text{air}}}}$$

Free falling sphere

$$F_{res} = F_g - F_n \implies F_g = F_n$$

$$mg = C_D A \rho_{\text{fluid}} \frac{v_\infty^2}{2} \implies v_\infty = \sqrt{\frac{2mg}{C_D A \rho_{\text{fluid}}}}$$

Velocity profile at the initial phase ($F_{res} \neq 0$)

$$m\frac{dv}{dt} = mg - C_D A \rho_{\text{fluid}} \frac{v^2}{2}$$

$$\int_0^t dt = \int_0^v \frac{m}{mg - C_D A \rho_{\text{fluid}} \frac{v^2}{2}} dv$$

$$v(t) = v_\infty \tanh \frac{gt}{v_\infty}$$

Duration of the initial phase

If steady terminal velocity $v(t) = v_\infty \cdot 0.99$:

$$\tanh \frac{gt}{v_\infty} = 0.99 \implies \frac{gt}{v_\infty} = \text{arctanh}(0.99)$$

$$t = \frac{v_\infty \, \text{arctanh}(0.99)}{g} = \frac{v_\infty \cdot 2.65}{g}$$

Lift-induced drag and Parasitic drag

$$C_D = C_{D_0} + C_{D_i}$$

$$C_{D_i} = \frac{C_L^2}{\pi e AR} \quad ; \quad AR = \frac{b^2}{A}$$

- low speed → induced drag dominates
- high speed → parasitic drag dominates

Parasitic drag

Drag not due to lift generation (skin friction + form drag + interference drag).

$$F_{D_0} = C_{D_0} \frac{1}{2} \rho v_\infty^2 A$$

Lift-induced drag

Drag due to lift generation, caused by wing-tip vortices and downwash.

$$F_{D_i} = C_{D_i} \frac{1}{2} \rho v_\infty^2 A$$

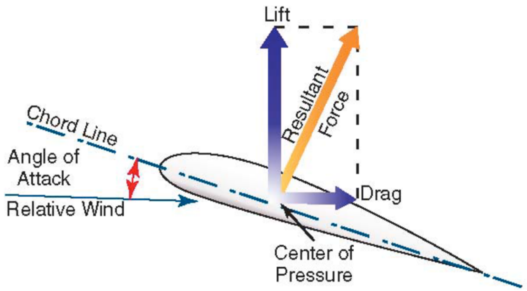
For constant lift:

$$F_{D_0} \sim v_\infty^2 \quad ; \quad F_{D_i} \sim \frac{1}{v_\infty^2}$$

Lift force and lift coefficient

Angle of Attack (AOA)

Variable angle between the chord line and the relative wind; affects lift and can change during flight.

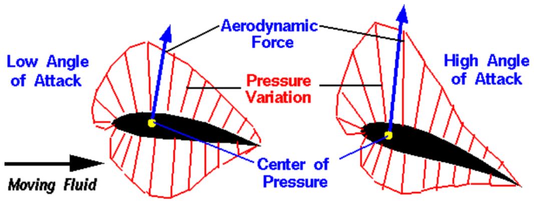


Angle of Incidence (AOI)

Fixed angle between the chord line and the aircraft's longitudinal axis; set during design and affects baseline performance

Center of pressure

Centre of pressure is the resultant of the pressure distribution on an airfoil



NACA Series

4-Digit-Series

E.g. NACA 2421

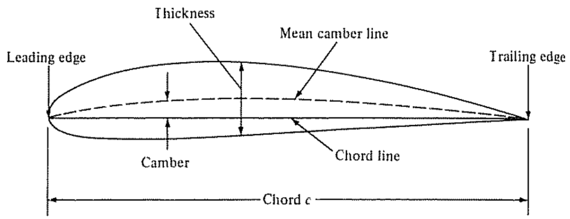
- 1st Digit: Maximum camber is 2% of airfoil chord length c
- 2nd Digit: Location of maximum camber is 4/10ths (40%) of chord length
- 3rd & 4th Digits: Maximum Thickness is 21% of chord length

5-Digit-Series

E.g. NACA 23012

- 1st Digit: Design lift coefficient
  - $c_L = 2 * \frac{3}{20} = 0.3$
- 2nd & 3rd: Position of max camber
  - 3: Row 30 in NACA table
  - 0/1: Standard or Reflex
- 4th&5th Digit: Maximum thickness is 12% of total chord length

Airfoil terminology



IV. Dimensional Analysis and Similarity

Dimensional homogeneity

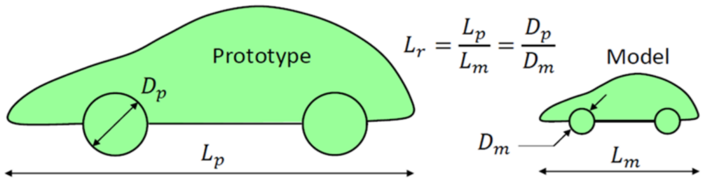
Name	Variable	Primary Dimension
Length	l, d	L
Time	t	T
Mass	m	M
Temperature	T	Θ
Molar amount	n	N
Velocity	u, v, w	LT^{-1}
Rotational Speed	s	T^{-1}
Acceleration	a, g	LT^{-2}
Kinematic Viscosity	ν	L^2T^{-1}
Diffusion coefficient	D	L^2T^{-1}
Force	F	$LT^{-2}M$
Energy	E	$L^2T^{-2}M$
Work, Heat	W, Q	$L^2T^{-2}M$
Power	P	$L^2T^{-3}M$
Pressure	p	$L^{-1}T^{-2}M$
Impuls	I	$LT^{-1}M$
Surface Tension	γ, σ	$T^{-2}M$
Density	ρ	$L^{-3}M$
Dynamic Viscosity	η	$L^{-1}T^{-1}M$
Heat Capacity	C	$L^2T^{-2}M\Theta^{-1}$
Molar heat capacity	c	$L^2T^{-2}M\Theta^{-1}N^{-1}$

Similarity

Three basic laws of similarity must be satisfied in order to achieve complete similarity between prototype and model flow fields

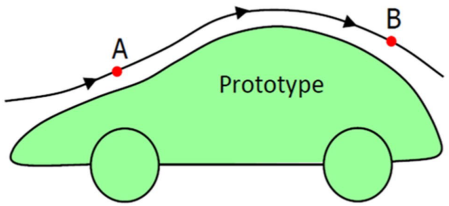
Geometric similarity

- Model and prototype have same shape
- Linear dimensions on model and prototype correspond within constant scale factor

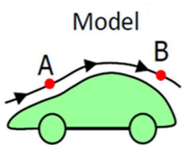


Kinematic similarity

Velocities at corresponding points on model and prototype differ only by a constant scale factor



$$V_r = \frac{V_{p_A}}{V_{m_A}} = \frac{V_{p_B}}{V_{m_B}}$$



Dynamic similarity

All forces on model and prototype differ only by a constant scale factor

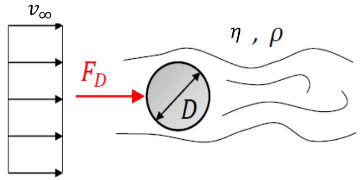
Buckingham Π-Theorem

Buckingham Pi-Theorem can be used to determine the nondimensional groups of variables (Pi-Groups) for a given set of dimensional variables.

Step 1. List all the dimensional variables involved (n)
Step 2. Express each variables using primary dimensions (r):
- Only L: **geometric variables**
- Only T or both L and T: **kinematic variables**
- Those who have M: **dynamic variables**

Step 3. Determine the repeating variables that are allowed to appear in more than one Pi-group
Step 4. Determine (n-r) many Pi-Groups by combining repeating variables with non-repeating variables and using the fact that Pi-Groups should be non-dimensional

Example with drag force



Step 1. $n = (F_D, D, v_\infty, \eta, \rho) = 5$

Step 2. $F_D = \frac{ML}{T^2}; \quad D = L; \quad v_\infty = \frac{L}{T}; \quad \rho = \frac{M}{L^3}; \quad \eta = \frac{M}{LT}$
$r = [D, v_\infty, (F_D, \eta, \rho)] = 3$

Step 3. Repeating variables: $D\ [L], v_\infty\ [T], \rho\ [M]$

Step 4. Number of Π-groups: $n - r = 5 - 3 = 2$
$\Pi_1 = F_D D^{a_1} v_\infty^{b_1} \rho^{c_1} \quad ; \quad \Pi_2 = \eta D^{a_2} v_\infty^{b_2} \rho^{c_2}$

Nondimensional numbers

Euler Number

$$Eu = \frac{\text{Pressure energy}}{\text{Kinetic energy}} = \frac{\Delta p}{\rho v^2} = \frac{gH}{v^2}$$

Important for:
- Nozzle and Diffuser
- Pipe and Ducts
- Pumps and Turbines

Grasshof Number

$$Gr = \frac{\text{Buoyancy force}}{\text{Viscous forces}} = \frac{L^3 g \beta \Delta T}{\nu}$$

Important for:
- **Natural (Free) Convection**
- $Gr \ll 1$: Conduction dominates
- $Gr \gg 1$: Convection dominates
- Radiators and heaters
- Cooling of electronics (no Fan)

Froude Number

$$\frac{\text{Inertia forces}}{\text{Gravitational force}} = \frac{v}{\sqrt{gL}}$$

Important for:
- Ship Design: Controls wave formation and resistance
- General water flows
- Open channel / Free Surface flows
- $Fr \ll 1$: Gravity has high influence
- $Fr \gg 1$: Gravity negligible

Cavitation Number

$$Ca = \sigma = \frac{p_\infty - p_v}{\frac{1}{2}\rho v^2}$$

Important for:
- Pump Design, Ship Propeller Design
- Describes savety margin to cavitation:
  $\sigma \approx 1$: Pressure close to vapor pressure

Weber Number

$$We = \frac{\text{Inertia forces}}{\text{Surface tension forces}} = \frac{\rho v L^2}{\sigma}$$

Important for:
- Fuel injection
- Jet breakup
- Bubble formation

Prandlt Number

$$Pr = \frac{\text{Viscous diffusion}}{\text{Thermal diffusivity}} = \frac{c_p \nu}{\lambda} = \frac{\nu}{\alpha}$$

- $Pr < 1$: Heat diffuses quickly compared to viscous (momentum) diffusion
- $Pr = 1$: Momentum BL $\sim$ Temperature BL
- $Pr > 1$: Heat is transported through diffusion in momentum

Nusselt Number (Forced Convection)

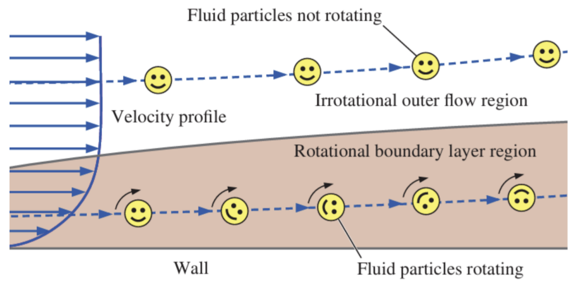
Nusselt number is to the thermal boundary layer what the friction coefficient is to the velocity boundary layer:

$$Nu = \frac{\alpha L}{\lambda} \approx \frac{\text{Convective heat transfer}}{\text{Conductive heat transfer}} = f(x^*, Re_L, Pr)$$

Regime	Velocity BL	Thermal BL	$Nu(x)$
Laminar	$\delta(x) = x \cdot \frac{4.91}{\sqrt{Re_x}}$	$\frac{\delta_t}{\delta} \approx Pr^{-1/3}$	$0.332\, Re^{\frac{1}{2}}\, Pr^{0.7}$
Turbulent	$\delta(x) = x \cdot \frac{0.16}{Re_x^{1/7}}$	$\frac{\delta_t}{\delta} \approx Pr^{-1/3}$	$0.023\, Re^{0.8}\, Pr^{1/3}$

V. Potential flow

Irrotationality



$$\tau = \eta \frac{du}{dy} = 0$$

Vorticity

2D flow

For fluids with constant viscosity, a 2D flow is irrotational when:

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

3D flow

In 3D, the velocity becomes a vector:

$$\underline{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\underline{\omega} = \text{rot}(\underline{u}) = \nabla \times \underline{u} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Cartesian system

$$\underline{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \text{ and } \nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \Rightarrow \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{pmatrix}$$

Cylindrical system

$$\underline{u} = \begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix} \text{ and } \nabla = \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial z} \end{pmatrix} \Rightarrow \begin{pmatrix} \omega_r \\ \omega_\theta \\ \omega_z \end{pmatrix} = \begin{pmatrix} \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \\ \frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta} \end{pmatrix}$$

Potential fields

$$\text{rot}(\underline{u}) = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

$$\underline{u} = \nabla(\Phi) \Longrightarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \\ \frac{\partial \Phi}{\partial z} \end{pmatrix}$$

$$w_z = \frac{\partial^2 \Phi}{\partial x \partial y} - \frac{\partial^2 \Phi}{\partial y \partial x} = 0$$

General fluid motion

Governing equations

Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \underline{u} = 0$$

Momentum equation

$$\frac{\partial \rho \underline{u}}{\partial t} + \nabla \cdot \rho \underline{u}\, \underline{u} = \nabla p + \eta \nabla \cdot \underline{\tau}$$

Simplifications

- Inviscid: $\eta = 0$
- Steady: $\frac{\partial}{\partial t} = 0$
- Irrotational: $\text{rot}(\underline{u}) = 0 \to \underline{u} = \nabla \Phi$
- Incompressible: $\rho = ¢$
- Continuity: $\nabla \cdot \underline{u} = 0$

Potential flow model

Velocity potential: $\underline{u} = \nabla \Phi$

Laplace governing equation: $\nabla^2 \Phi = 0$

Pressure field (Euler → Bernoulli): $p + \frac{1}{2}\rho u^2 = ¢$

Potential function $\phi(x, y)$

Scalar potential field and Potential flow

Incompressible, steady flow and irrotational:

$$\zeta = \text{rot}(u) = \nabla \times u = 0$$

Hence:

$$u = \nabla \Phi = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{\nabla \Phi}{\partial x} \\ \frac{\nabla \Phi}{\partial y} \end{pmatrix}$$

Continuity equation

$$\nabla \cdot u = 0 \to \nabla \cdot \nabla \Phi = \nabla^2 \Phi = 0$$

Momentum (Euler) equation

$$u \cdot \nabla u = \nabla \frac{1}{2}|u|^2\, u \cdot \nabla u = \nabla \frac{1}{2}|u|^2 - u \times (\nabla \times u)$$

which reduces to:

$$\nabla \frac{1}{2}|u|^2 + \frac{p}{\rho} = ¢$$

Potential vortex

Assuming $\eta = 0$ and irrotational flow:

$$u_r(r, \theta) = 0 \quad ; \quad u_\theta(r, \theta) = v(r)$$

Velocity distribution

$$v_\theta(r) = \frac{C_\Phi}{r} = \frac{\Omega R^2}{r} = \frac{\Gamma}{2\pi r}$$

$$\Gamma = \oint v(r)\, ds = 2\pi r v(r) = 2\pi \Phi$$

where Γ is the circulation.

Volume over a radial plane

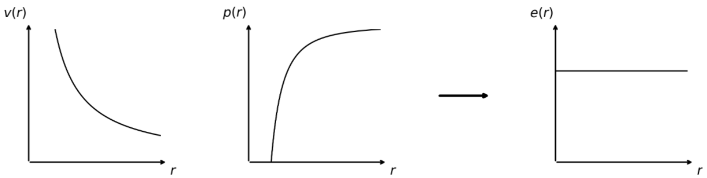
$$\dot{V} = \int_{R_1}^{R_2} v_\theta(r)\, dA = \int_{R_1}^{R_2} \frac{\Omega R^2}{r} h\, dr = \Omega R^2 h \ln\left(\frac{R_2}{R_1}\right)$$

Pressure distribution

The total pressure is constant throughout the whole field

$$p(r) = p_{\text{ref}} + \frac{\rho}{2}\left(v_{\text{ref}}^2 - v(r)^2\right) = ¢$$

Velocity, pressure, and energy graph



Proof of irrotationability

$$\vec{\omega} = \nabla \times \vec{u} = \begin{pmatrix} \omega_r \\ \omega_\theta \\ \omega_z \end{pmatrix} = \begin{pmatrix} \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta} \end{pmatrix}$$

For the potential vortex:

$$u_r = 0, \qquad u_\theta = \frac{C_\Phi}{r}, \qquad u_z = 0$$

Therefore:

$$\omega_r = \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} = 0 - 0 = 0$$

$$\omega_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} = 0 - 0 = 0$$

$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{C_\Phi}{r}\right) - 0 = \frac{1}{r}\frac{\partial C_\Phi}{\partial r} = 0$$

Thus:

$$\vec{\omega} = \nabla \times \vec{u} = \vec{0}$$

and the potential vortex is irrotational for $r \neq 0$.

Source and Sink

Source and Sink behave as an inverted Potential Vortex:

$$u_r(r,\theta) = v(r) \qquad ; \qquad u_\theta(r,\theta) = 0$$

Velocity distribution

$$v(r) = \frac{C_r}{r} = \frac{E}{2\pi r}$$

$$V = \int v(r)\,dA = 2\pi l_u v(r) = \frac{E}{l_z}$$

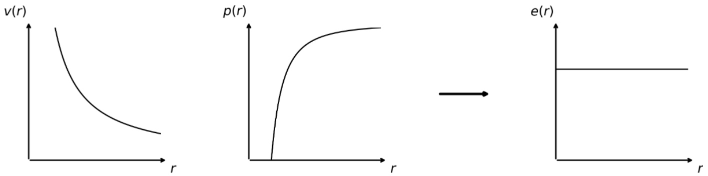
where E is called Strength

Pressure distribution

The total pressure is constant throughout the whole field

$$p(r) = p_{\text{ref}} + \frac{\rho}{2}\left(v_{\text{ref}}^2 - v(r)^2\right) = \not{c}$$

Velocity, pressure, and energy graph



Rigid body vortex

Fluid rotates like a rigid body with angular velocity Ω

$$u_r(r,\theta) = 0 \quad ; \quad u_\theta(r,\theta) = u_\theta(r)$$

Velocity distribution

$$v(r) = \Omega r$$

Angular velocity distribution

$$\omega_z = 2\Omega$$

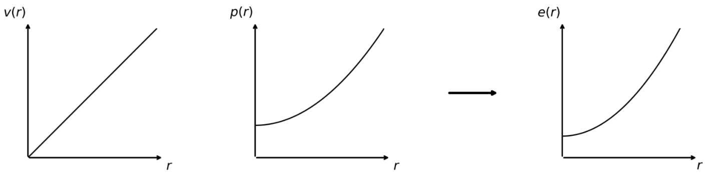
Pressure distribution

$$p(r) = p_{\text{ref}} + \frac{1}{2}\rho\Omega^2\left(r^2 - r_{\text{ref}}^2\right)$$

Total pressure increase with the radius

$$p_t(r) = p(r) + \frac{1}{2}\rho v(r)^2 = p(r) + \frac{1}{2}\rho\Omega^2 r^2$$

Velocity, pressure, and energy graph



Superposition: Potential Vortex and Sink / Source

Velocity distribution

$$\underline{u} = \begin{pmatrix} v_r \\ v_\varphi \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{C_r}{r} \\ 0 \end{pmatrix}}_{\text{Sink}} + \underbrace{\begin{pmatrix} 0 \\ \frac{C_\varphi}{r} \end{pmatrix}}_{\text{Potential Vortex}} = \begin{pmatrix} \frac{C_r}{r} \\ \frac{C_\varphi}{r} \end{pmatrix}$$

$$\frac{v_r}{v_\varphi} = \frac{C_r}{C_\varphi} = \tan(\alpha) = C$$

The magnitude of the velocity:

$$v = \sqrt{v_\varphi^2 + v_r^2} = \sqrt{\frac{C_\varphi^2 + C_r^2}{r^2}} = \frac{K}{r}$$

$$v(r) = v_{\text{ref}}\frac{r_{\text{ref}}}{r}$$

Streamlines

Streamline trajectory

$$\tan(\alpha) = \frac{dr}{r\,d\varphi}$$

$$\tan(\alpha)\,d\varphi = \frac{1}{r}\,dr$$

Integrating with respect to (r_0, φ_0) and (r, φ):

$$\tan(\alpha)\int_{\varphi_0}^{\varphi} d\varphi = \int_{r_0}^{r}\frac{1}{r}\,dr$$

$$\tan(\alpha)\cdot(\varphi - \varphi_0) = \ln\left(\frac{r}{r_0}\right)$$

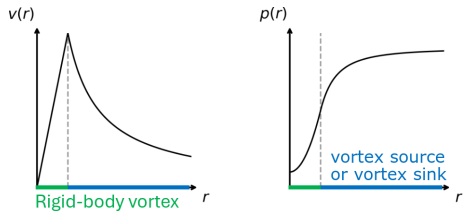
Streamline angle

$$\varphi = \frac{1}{\tan(\alpha)}\ln\left(\frac{r}{r_0}\right) + \varphi_0$$

Pressure distribution

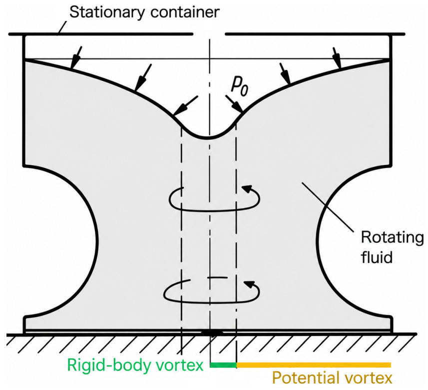
$$p(r) = p_{\text{ref}} + \frac{1}{2}\rho v_{\text{ref}}^2\left(1 - \left(\frac{r_{\text{ref}}}{r}\right)^2\right)$$

Velocity and pressure graph



Superposition: Rankine vortex

Potential vortex + Rigid body vortex = Rankine vortex



Piecewise velocity distribution

Inner region ($r < R_0$) – Rigid body vortex

$$u_r = 0 \quad ; \quad u_\theta = \Omega r$$

Constant velocity:

$$\omega_z = \frac{1}{r}\frac{d\,(r u_\theta)}{dr} = 2\Omega$$

Outer region ($r > R_0$) – Potential vortex

$$u_r = 0 \quad ; \quad u_\theta = \frac{\Gamma}{2\pi r}$$

No vorticity:

$$\omega_z = \frac{1}{r}\frac{d\,(r u_\theta)}{dr} = 0$$

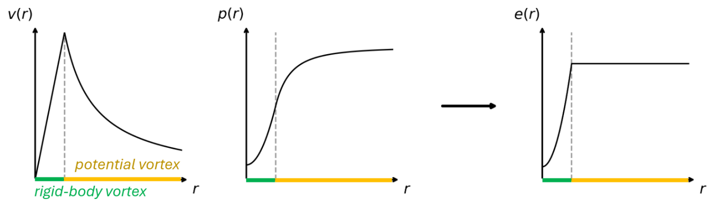
Transition region ($r = R_0$) – Same velocity

$$\Omega r = \frac{\Gamma}{2\pi r}$$

Coupling between circulation and angular velocity:

$$\Gamma = 2\pi\Omega R_0^2$$

Velocity, pressure, and energy graph



VI. Compressible flow

Incompressible vs compressible flows

Incompressible
- Mass conservation: ρ
- Momentum conservation: $\vec{u}$
- Energy conservation: $h \to T$

Compressible
- Mass conservation: ρ
- Momentum conservation: $\vec{u}$
- Energy conservation: $h \to T$
- Eq. of state: $p = f(\rho, T)$

Compressibility and Mach number

Mach number

$$\boxed{M = \frac{v}{c}}$$

Mach number	Flow	Note
$M < 0.3$	Incompressible	Density changes are negligible
$0.3 < M < 0.9$	Subsonic	No shock wave, density effects are important
$0.9 < M < 1.1$	Transonic	Shock waves; subsonic and supersonic regions
$1.1 < M < 3.0$	Supersonic	Shock waves, no subsonic regions
$M > 3.0 - 5.0$	Hypersonic	Strong shock waves, property changes

Shock waves

Normal at 90° (perpendicular) to the flow direction

Oblique inclined to the flow direction

Bowed Forms ahead of a blunt body when the upstream flow is supersonic

Oblique shock

- θ: turning / deflection angle
- β: shock / wave angle
- No boundary layer growth: $\theta = \delta$
- Conservation of mass:

$$\rho_1 V_{1,n} A = \rho_2 V_{2,n} A \quad \Longrightarrow \quad \rho_1 V_{1,n} = \rho_2 V_{2,n}$$

- Conservation of momentum:

$$p_1 - p_2 = \rho_2 V_{2,n}^2 - \rho_1 V_{1,n}^2$$

- Normal Mach numbers:

$$M_{1,n} = M_1 \sin\beta, \qquad M_{2,n} = M_2 \sin(\beta - \theta)$$

- For oblique shock, only the normal component becomes subsonic.

Speed of sound

$$\boxed{c = \sqrt{\left.\frac{dp}{d\rho}\right|_s} = \sqrt{kR_sT}}$$

where $k = \dfrac{c_p}{c_v}$

Thermodynamics

Internal energy

$$c_v = \left.\frac{\partial u}{\partial T}\right|_V \to du = c_v(T)\,dT$$

$$u_2 - u_1 = \int_{T_1}^{T_2} c_v(T)\,dT = c_v \Delta T = c_v\,(T_2 - T_1)$$

Enthalpy

$$h = u + pv = u + R_s T$$

$$c_p = \left.\frac{\partial h}{\partial T}\right|_p \to dh = c_p(T)\,dT$$

$$h_2 - h_1 = \int_{T_1}^{T_2} c_p(T)\,dT = c_p \Delta T = c_p\,(T_2 - T_1)$$

Specific heat capacity

$$c_p - c_v = \frac{dh}{dT} - \frac{du}{dT} = R_s$$

$$k = \frac{c_p}{c_v} \quad ; \quad c_p = \frac{R_S k}{k-1} \quad ; \quad c_v = \frac{R_S}{k-1}$$

Density

$$pV = mR_iT \Longrightarrow p = \rho R_i T \Longrightarrow \rho = \frac{p}{R_i T}$$

Entropy

$$T\,ds = du + p\,dv \quad ; \quad T\,ds = dh - v\,dp$$

Ideal gas

$$s_2 - s_1 = c_v \ln\left(\frac{T_2}{T_1}\right) + R_s \ln\left(\frac{\rho_1}{\rho_2}\right)$$

$$= c_p \ln\left(\frac{T_2}{T_1}\right) - R_s \ln\left(\frac{p_2}{p_1}\right)$$

Isentropic flow

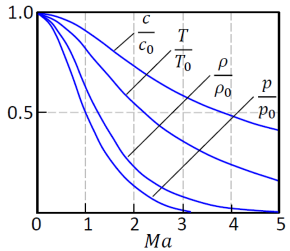
$$\left(\frac{T_2}{T_1}\right)^{\frac{k}{k-1}} = \left(\frac{\rho_2}{\rho_1}\right)^k = \frac{p_2}{p_1}$$

Stagnation state and Mach number

$$T_0 = T + \frac{v^2}{2c_p} \Longleftrightarrow \frac{T_0}{T} = 1 + \frac{v^2}{2c_p \cdot T}$$

$$c_p = \frac{k}{k-1}R \Longleftrightarrow \frac{1}{c_p} = \frac{k-1}{k \cdot R}$$

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}\frac{v^2}{c^2} = 1 + \frac{k-1}{2}M^2$$



Pressure ratio

$$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{k}{k-1}} = \left(1 + \frac{k-1}{2}M^2\right)^{\frac{k}{k-1}}$$

Density ratio

$$\frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{\frac{1}{k-1}} = \left(1 + \frac{k-1}{2}M^2\right)^{\frac{1}{k-1}}$$

Critical state $M = 1$

$$\frac{p_0}{p^*} = \left(1 + \frac{k-1}{2}\right)^{\frac{k}{k-1}}$$

$$\frac{T_0}{T^*} = 1 + \frac{k-1}{2}$$

$$\frac{\rho_0}{\rho^*} = \left(1 + \frac{k-1}{2}\right)^{\frac{1}{k-1}}$$

$$\frac{A}{A^*} = \frac{1}{M}\left(\frac{1 + \frac{k-1}{2}M^2}{\frac{k+1}{2}}\right)^{\frac{k+1}{2(k-1)}}$$

Isentropic flow with area change

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dv}{v} = 0$$

$$\frac{dA}{A} = \frac{dp}{\rho}\left(\frac{1}{v^2} - \frac{d\rho}{dp}\right)$$

$$\frac{dA}{A} = \frac{dp}{\rho v^2}\left(1 - M^2\right) = -\frac{dv}{v}\left(M^2 - 1\right)$$

Nozzle results

Subsonic:	$M < 1$	$dp\,(1 - M^2) < 0$	$dA < 0$
Sonic:	$M = 1$	$dp\,(1 - M^2) = 0$	$dA = 0$
Supersonic:	$M > 1$	$dp\,(1 - M^2) > 0$	$dA > 0$

Diffuser results

Subsonic: $M<1 \quad dp\left(1-M^2\right)>0 \quad dA>0$

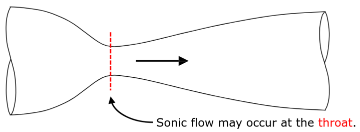
Sonic: $M=1 \quad dp\left(1-M^2\right)=0 \quad dA=0$

Supersonic: $M>1 \quad dp\left(1-M^2\right)<0 \quad dA<0$

Nozzle and diffuser summary

	Subsonic $(Ma<1)$	Supersonic $(Ma>1)$
Diffuser $dp>0 \;;\; dv<0$	$dA>0$	$dA<0$
Nozzle $dp<0 \;;\; v>0$	$dA<0$	$dA>0$

Equation of mass flow rate through a nozzle



Putting $\dfrac{d\dot m}{d\left(\frac{p}{p_0}\right)}=0$:

$$\dot m = A\cdot p_0\sqrt{\frac{2k}{(k-1)R_S T_0}}\cdot\sqrt{\left(\frac{p}{p_0}\right)^{\frac{2}{k}}-\left(\frac{p}{p_0}\right)^{\frac{k+1}{k}}}$$

This leads to

$$\frac{p}{p_0}=\left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}=\frac{p^*}{p_0}$$

Chocked nozzle

Using $\dfrac{p_0}{p}=\left(1+\dfrac{k-1}{2}M^2\right)^{\frac{k}{k-1}}$:

$$\dot m=\sqrt{\frac{k}{R_S T_0}}\cdot\frac{A\cdot p_0\cdot M}{\left(1+\frac{k-1}{2}M^2\right)^{\frac{k+1}{2(k-1)}}}$$

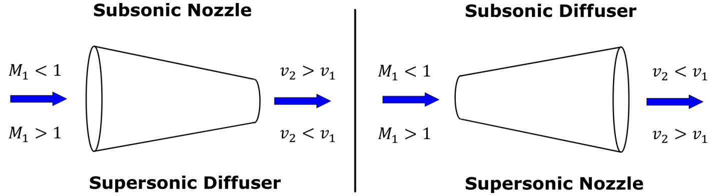
When $M=1,\ A=A^*$, hence

$$\dot m = A^*\cdot p_0\sqrt{\frac{k}{R_S T_0}}\left(\frac{2}{k+1}\right)^{\frac{k+1}{2(k-1)}}$$

The ratio of the area:

$$\frac{A}{A^*}=\frac{1}{M}\left[\left(\frac{2}{k+1}\right)\left(1+\frac{k-1}{2}M^2\right)\right]^{\frac{k+1}{2(k-1)}}$$

Nozzle - Diffuser



VII. EFPLab 1

Fluid state variables

Kinematic viscosity

$$\nu := \frac{\eta}{\rho}\left[\frac{m^2}{s}\right]$$

Dynamic viscosity

$$\eta := \nu\cdot\rho\left[Pa\cdot s=\frac{Ns}{m^2}=\frac{kg}{m\cdot s}\right]$$

Pressure

$$p_{\text{tot}}=p_{\text{stat}}+p_{\text{dyn}}=\rho\left(gh+\frac{v^2}{2}\right)$$

$$p_{\text{rel}}=p_{\text{abs}}-p_{\text{ref}}\Longleftrightarrow p_{\text{abs}}=p_{\text{rel}}+p_{\text{ref}}\geq 0$$

Hydrostatic

$$p=p_0+\rho g h=¢\ [Pa]$$

$$p=p_0+\frac{F_g}{A}=p_0+\frac{mg}{A}=p_0+\frac{\rho h A g}{A}$$

Mass conservation (incompressible fluid)

$$\dot m=\rho\dot V$$

$$\dot m_\alpha=\dot m_\omega\Longleftrightarrow\dot V_\alpha=\dot V_\omega$$

$$\dot V=\bar v A\Longrightarrow(\bar v A)_\alpha=(\bar v A)_\omega$$

Energy conservation

$$\dot m_1\left(\frac{p_1}{\rho}+\frac{v_1^2}{2}+gz_1\right)=\dot m_2\left(\frac{p_2}{\rho}+\frac{v_2^2}{2}+gz_2\right)$$

Energy flow

$$\frac{dE}{dt}=\sum P+\sum Q+\sum_\alpha\dot m\left(\frac{p}{\rho}+\frac{v^2}{2}+gz\right)-\sum_\omega\dot m\left(\frac{p}{\rho}+\frac{v^2}{2}+gz\right)$$

Bernoulli equation

Specific energy equation

$$\frac{p_1}{\rho}+\frac{v_1^2}{2}+gz_1=\frac{p_2}{\rho}+\frac{v_2^2}{2}+gz_2=¢\left[\frac{J}{kg}\right]$$

Pressure equation

$$p_1+\frac{\rho v_1^2}{2}+\rho g z_1=p_2+\frac{\rho v_2^2}{2}+\rho g z_2=¢\ [Pa]$$

Height equation

$$\frac{p_1}{\rho g}+\frac{v_1^2}{2g}+z_1=\frac{p_2}{\rho g}+\frac{v_2^2}{2g}+z_2=¢\ [m]$$

Extended Bernoulli equation

$$\frac{p_1}{\rho}+\frac{v_1^2}{2}+gz_1+e_A=\frac{p_2}{\rho}+\frac{v_2^2}{2}+gz_2+e_V\left[\frac{J}{kg}\right]$$

Work terms

$$e_A=\frac{p_A}{\rho}=gz_A=\frac{E_A}{m}=\frac{P_A}{\dot m}\left[\frac{J}{kg}\right]$$

$$e_V=\frac{p_V}{\rho}=gz_V=\frac{E_V}{m}=\frac{P_V}{\dot m}\left[\frac{J}{kg}\right]$$

Pump and turbine work Y

$$e_w=Y=\frac{W_A}{\dot m}=\frac{E_A}{m}=H\cdot g=\frac{p_A}{\rho}\left[\frac{J}{kg}\right]$$

$$P_{\text{hyd}}=\dot m\cdot Y=\rho\dot V Y=\rho\dot V g H\ [W]$$

Pressure loss

$$\Delta p_V=e_V\cdot\rho=\frac{E_V\cdot\rho}{m}=\rho g z_V=\zeta\cdot\rho\frac{v^2}{2}\ [Pa]$$

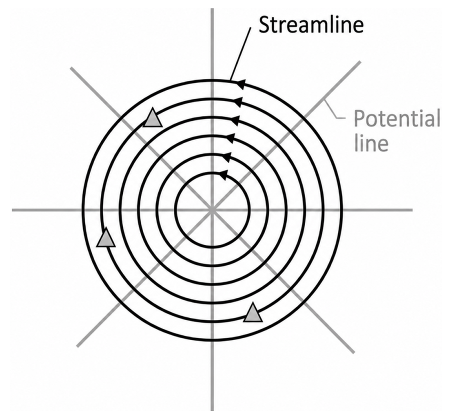
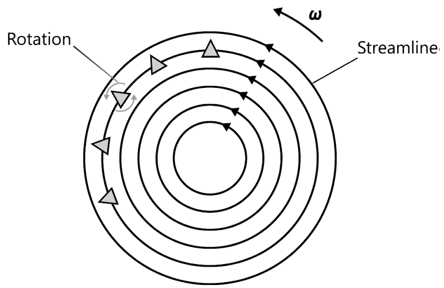
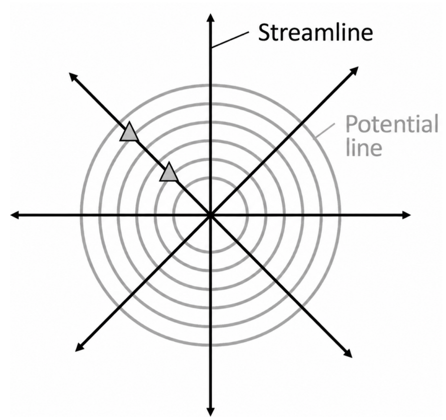
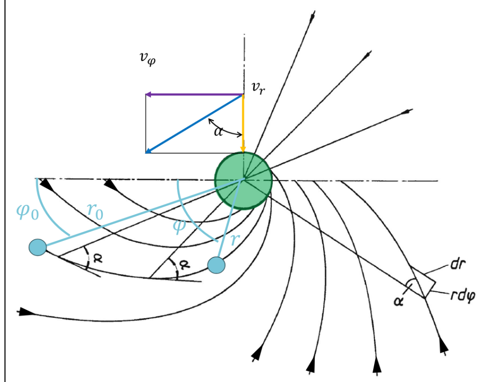
where $\zeta=\dfrac{2\Delta p_V}{\rho v^2}$

Pressure head

$$H=\frac{e_A}{g}=\frac{\Delta p_A}{\rho g}\ [m]$$

U-tube manometer

$$h=\frac{\rho\left(v_2^2-v_1^2\right)}{2g\left(\rho_{\text{fl,u-tube}}-\rho_{\text{fl,pipe}}\right)}$$

	1. POTENTIAL VORTEX	2. RIGID-BODY VORTEX	3. SOURCE / SINK	4. VORTEX + SOURCE/SINK (OVERLAY)
Flow pattern (streamlines)				
Velocity components	$v_r = 0, \qquad v_\varphi = \frac{C_\varphi}{r}$	$v_r = 0, \qquad v_\varphi = \Omega r$	$v_r = \frac{C_r}{r}, \qquad v_\varphi = 0$	$v_r = \frac{C_r}{r}, \qquad v_\varphi = \frac{C_\varphi}{r}$
Speed magnitude $v(r)$	$v(r) = \frac{\lvert C_\varphi\rvert}{r}$	$v(r) = \lvert\Omega\rvert\, r$	$v(r) = \frac{\lvert C_r\rvert}{r}$	$v(r) = \frac{\sqrt{C_r^2 + C_\varphi^2}}{r}$
Flow angle α **(defined from** v_r, v_φ**)**	$\alpha = 90^\circ$ (purely tangential)	$\alpha = 90^\circ$ (purely tangential)	$\alpha = 0^\circ$ (purely radial outward for $C_r > 0$, inward for $C_r < 0$)	$\tan\alpha = \frac{v_r}{v_\varphi} = \frac{C_r}{C_\varphi} = \text{const.}$ $\Rightarrow \alpha = \arctan\left(\frac{C_r}{C_\varphi}\right)$
Pressure $p(r)$	$p(r) = p_{\text{ref}} + \frac{\rho}{2}\left(v_{\text{ref}}^2 - \frac{C_\varphi^2}{r^2}\right)$	$p(r) = p_{\text{ref}} + \frac{1}{2}\rho\left(u_\theta^2(r) - u_\theta^2(r_{\text{ref}})\right)$	$p(r) = p_{\text{ref}} + \frac{\rho}{2}\left(v_{\text{ref}}^2 - \frac{C_r^2}{r^2}\right)$	$p(r) = p_{\text{ref}} + \frac{\rho}{2}\left(v_{\text{ref}}^2 - v(r)^2\right)$ $= p_{\text{ref}} + \frac{\rho}{2}\left(v_{\text{ref}}^2 - \frac{C_r^2 + C_\varphi^2}{r^2}\right)$
Properties	• Irrotational for $r \neq 0$ • Singular at $r = 0$ • Circular streamlines • Circulation $\Gamma = 2\pi C_\varphi$	• Rotational (solid-body rotation) • Angular velocity $\Omega = v_\varphi / r =$ const. • Finite at center ($v = 0$ at $r = 0$) • Models viscous core	• Irrotational for $r \neq 0$ • Singular at $r = 0$ • Radial streamlines • Source: $C_r > 0$ Sink: $C_r < 0$	• Irrotational for $r \neq 0$ • Singular at $r = 0$ • Spiral (logarithmic) streamlines • Flow angle α constant • Reduces to cases 1 or 3 if C_r or $C_\varphi = 0$
Circulation Γ	$\boldsymbol{\Gamma = 2\pi C_\varphi}$	$\boldsymbol{\Gamma(r) = 2\pi\Omega r^2}$	$\boldsymbol{\Gamma = 0}$	$\boldsymbol{\Gamma = 2\pi C_\varphi}$